

PARABOLIC HIGGS BUNDLES AND REPRESENTATIONS OF THE FUNDAMENTAL GROUP OF A PUNCTURED SURFACE INTO A REAL GROUP

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ABSTRACT. We study parabolic G -Higgs bundles over a compact Riemann surface with fixed punctures, when G is a real reductive Lie group, and establish a correspondence between these objects and representations of the fundamental group of the punctured surface in G with arbitrary holonomy around the punctures. Three interesting features are the relation between the parabolic degree and the geometry of the Tits boundary, the treatment of the case when the logarithm of the monodromy is on the boundary of a Weyl alcove, and the correspondence of the orbits encoding the singularity via the Kostant–Sekiguchi correspondence.

Dedicated with admiration and gratitude to Narasimhan and Seshadri in the fiftieth anniversary of their theorem

1. INTRODUCTION

The relation between representations of the fundamental group of a compact Riemann surface X into a compact Lie group, and holomorphic bundles on X goes back to the celebrated theorem of Narasimhan and Seshadri [37], where they established a homeomorphism between the moduli space of irreducible representation of $\pi_1(X)$ in the unitary group $U(n)$ and the moduli space of rank n and zero degree stable holomorphic vector bundles on X . Of course, this generalises the classical case of representations in $U(1)$ and their relation with the Jacobian of X . The Narasimhan–Seshadri theorem has been a paradigm and an inspiration for more than 50 years now for many similar problems. The theorem was generalised by Ramanathan [38] to representations into any compact Lie group [38]. The gauge-theoretic point of view of Atiyah and Bott [1], and the new proof of the Narasimhan–Seshadri theorem given by Donaldson following this approach [17], brought new insight and new analytic tools into the problem.

The case of representations into a non-compact reductive Lie group G required the introduction of new holomorphic objects on the Riemann surface X called G -Higgs bundles. These were introduced by Hitchin [23, 24], who established a homeomorphism between the moduli space of reductive representation in $SL(2, \mathbb{C})$ and polystable $SL(2, \mathbb{C})$ -Higgs bundles. This correspondence was generalised by Simpson to any complex reductive Lie group (and in fact, to higher dimensional Kähler manifolds) [41, 43]. The correspondence in this case needed an extra ingredient — not present in the compact case — having to do with the existence of twisted harmonic maps into the symmetric space defined by G . This theorem was provided by Donaldson for $SL(2, \mathbb{C})$ [18] and by Corlette [16] for arbitrary G . In fact Corlette’s theorem, which holds for any reductive real Lie group, can be combined with an existence theorem for solutions to the Hitchin equations for a

G -Higgs bundle, given by the second and third authors in collaboration with Bradlow and Gothen [14, 19], to prove the correspondence for any real reductive Lie group G . In [43] Simpson gives an indirect proof of this by embedding G in its complexification when it exists.

There is another direction in which the Narasimhan–Seshadri theorem has been generalised. This is by allowing punctures in the Riemann surface. Here one is interested in studying representations of the fundamental group of the punctured surface with fixed holonomy around the punctures. These representations now relate to the parabolic vector bundles introduced by Seshadri [39]. The correspondence in this case for $G = \mathrm{U}(n)$ was carried out by Mehta and Seshadri [32]. A differential geometric proof modelled on that of Donaldson for the parabolic case was given by the first author in [5]. The case of a general compact Lie group is studied in [3, 44, 2] under suitable conditions on the holonomy around the punctures. One of the main issues for general G is about the appropriate generalization of parabolic principal bundles.

The non-compactness in the group and in the surface can be combined to study representations of the fundamental group of a punctured surface into a non-compact reductive Lie group G . Simpson considered this situation when $G = \mathrm{GL}(n, \mathbb{C})$ in [42]. A new ingredient in his work is the study of filtered local systems. The aim of this paper is to extend this correspondence to the case of an arbitrary real reductive Lie group G (including the case in which G is complex). We establish a one-to-one correspondence between reductive representations of the fundamental group of a punctured surface X with arbitrary holonomy around the punctures and polystable parabolic G -Higgs bundles on X .

One of the main technical issues to prove our correspondence, already present in [3, 44, 2], lies in the definition of parabolic principal bundles. If G is a non-compact reductive Lie group and $H \subset G$ is a maximal compact subgroup, we need to define parabolic $H^{\mathbb{C}}$ -bundles. This involves a choice for each puncture of an element in a Weyl alcove of H . If the element is in the interior of the alcove everything goes smoothly, but if the element is in a wall of the alcove, its adjoint may have eigenvalues equal to ± 1 (as opposed to the elements in the interior, whose eigenvalues in absolute value are strictly smaller than 1), and this introduces complications in the definition of the objects, as well as in the analysis to prove our existence theorems. However, we give a suitable definition of parabolic G -Higgs bundle including the case in which the elements are in a ‘bad’ wall of the alcove, which is appropriate to carry on the analysis and to prove the correspondence with representations. Of course the need of including elements in the walls of the Weyl alcove is determined by the fact that we want to have totally arbitrary fixed holonomy (conjugacy classes) around the punctures. Our approach for the bad weights is rather pedestrian, using holomorphic bundles and gauge transformations between them which can have meromorphic singularities, so that to a representation corresponds not a single holomorphic bundle, but rather a class of holomorphic bundles equivalent under such meromorphic transformations. A more formal algebraic point of view is that of parahoric torsors [22, 11, 2], but we preferred to stick to a more concrete definition which is sufficient to state completely the correspondence. Our definition is also more natural from the differential geometry viewpoint which sees holomorphic bundles only outside the punctures, and defines the sheaves of sections at the singular points by growth conditions with respect to model metrics.

Our approach involves some features that we would like to point out. As in Simpson $\mathrm{GL}(n, \mathbb{C})$ -case [42], we need to consider a slight extension of representations to the more general notion of filtered G -local systems, what we call parabolic G -local system. The definition of parabolic degree for both, parabolic G -local systems and

parabolic G -Higgs bundles, involve the Tits geometry of the boundary at infinity of the symmetric spaces G/H and $H^{\mathbb{C}}/H$, respectively. Another new feature is given the fact that relating the data at the punctures for the representation and the parabolic Higgs bundle implies a relation between G -orbits in \mathfrak{g} (the Lie algebra of G) and $H^{\mathbb{C}}$ -orbits in $\mathfrak{m}^{\mathbb{C}}$, where $\mathfrak{m}^{\mathbb{C}}$ is the complexification of \mathfrak{m} given by the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, which in the case of nilpotent orbits is known as Kostant–Sekiguchi correspondence [40, 29, 30, 45]. For general orbits, this correspondence is proved in [4, 8].

We give now a brief description of the different sections of the paper. In Section 2 we define parabolic principal bundles and the main ingredients needed, like parabolic degree, etc. In Section 3 we do the same for parabolic G -Higgs bundles and define the stability criteria. Section 4 is one of the technical sections where we carried out the analysis to prove the correspondence between polystable parabolic G -Higgs bundles and solutions of the Hermite–Einstein or Hitchin equation. As pointed out above, one of the main difficulties here is in dealing with parabolic structures lying in a ‘bad’ wall of the Weyl alcove. In Section 5 we introduce the notion of parabolic G -local system, prove an existence theorem for harmonic reductions and establish the relation with parabolic G -Higgs bundles completing the correspondence.

A substantial amount of the content of this paper appeared in the notes of a course given by the first author at the CRM (Barcelona) in 2010 [9]. We apologize for having taken so long to produce this paper. We believe that the results in this paper are the starting point for applying Higgs bundle methods to the study of the topology of the moduli spaces of representations of the fundamental group of a punctured surface in non-compact reductive Lie groups. This has been extremely successful in the case of a compact surface. For punctured surfaces some partial results on representations in $U(p, q)$ were obtained in [21] making some genericity assumptions, although the relation between representations and parabolic Higgs bundles which constitutes the main result in this paper was only sketched there (see [21, Theorem 13.2]). The results in the present paper may also be used to transport the results on the topology of parabolic $U(2, 1)$ -Higgs bundles obtained in [31] to the context of the moduli space of representations of the fundamental group.

Our approach will also allow to extend the results of Hitchin [25] for split real forms to the punctured situation, where there is an abundance of spaces that we can call Hitchin–Teichmüller components, depending on the sort of holonomy around the punctures. Similarly we can extend the theory of maximal Higgs bundles [13, 20, 10] to the punctured set up. Another direction that can be developed from here is the inclusion of higher order poles in the Higgs field, relating to wild non-abelian Hodge theory. These are all problems on which we plan to come back and that should lead to interesting new features.

Finally, we mention that V. Balaji informed us that he has an ongoing related work with I. Biswas and Y. Pandey.

2. PARABOLIC PRINCIPAL BUNDLES

2.1. Definition of parabolic principal bundle. Let X be a compact connected Riemann surface and let $\{x_1, \dots, x_r\}$ be a finite set of different points of X . Let $D = x_1 + \dots + x_r$ be the corresponding effective divisor.

Let $H^{\mathbb{C}}$ be a reductive complex Lie group. We fix a maximal compact subgroup $H \subset H^{\mathbb{C}}$ and a maximal torus $T \subset H$ with Lie algebra \mathfrak{t} .

Let E be a holomorphic principal $H^{\mathbb{C}}$ -bundle over X .

If M is any set on which $H^\mathbb{C}$ acts on the left, we denote by $E(M)$ the twisted product $E \times_{H^\mathbb{C}} M$. If M is a vector space (resp. complex variety) and the action of $H^\mathbb{C}$ on M is linear (resp. holomorphic) then $E(M)$ is a vector bundle (resp. holomorphic fibration). We denote by $E(H^\mathbb{C})$, the $H^\mathbb{C}$ -fibration associated to E via the adjoint representation of $H^\mathbb{C}$ on itself. Recall that for any $x \in X$ the fibre $E(H^\mathbb{C})_x$ can be identified with the set of antiequivariant maps from E_x to $H^\mathbb{C}$:

$$(2.1) \quad E(H^\mathbb{C})_x = \{ \phi : E_x \rightarrow H^\mathbb{C} \mid \phi(eh) = h^{-1}\phi(e)h \quad \forall e \in E_x, h \in H^\mathbb{C} \}.$$

We fix an alcove $\mathcal{A} \subset \mathfrak{t}$ of H containing $0 \in \mathfrak{t}$ (see Appendix A). Let $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$ and let $P_{\alpha_i} \subset H^\mathbb{C}$ be the parabolic subgroup defined by α_i as in §D.1. By Proposition A.2, for $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$ the eigenvalues of $\text{ad}(\alpha_i)$ have an absolute value smaller or equal than 1, and \mathfrak{p}_{α_i} is the sum of the eigenspaces of $\text{ad}(\alpha_i)$ for nonpositive eigenvalues of $\text{ad} \alpha_i$. We shall distinguish the subalgebra $\mathfrak{p}_{\alpha_i}^1 \subset \mathfrak{p}_{\alpha_i}$ defined by

$$\mathfrak{p}_{\alpha_i}^1 = \ker(\text{ad}(\alpha_i) + 1),$$

and the associated unipotent group $P_{\alpha_i}^1 \subset P_{\alpha_i}$.

We distinguish the subset $\mathcal{A}' \subset \bar{\mathcal{A}}$ of elements α such that the eigenvalues of $\text{ad} \alpha$ have an absolute value smaller than 1. When $\alpha_i \in \sqrt{-1}\mathcal{A}'$, then $P_{\alpha_i}^1$ is trivial.

We define a **parabolic structure of weight α_i on E over a point x_i** as the choice of subgroups $Q_i^1 \subset Q_i \subset E(H^\mathbb{C})_{x_i}$ isomorphic to the pair $P_{\alpha_i}^1 \subset P_{\alpha_i}$. By this we mean that there exists some trivialization $e \in E_{x_i}$ for which $P_{\alpha_i} = \{ \phi(e) \mid \phi \in Q_i \}$ and $P_{\alpha_i}^1 = \{ \phi(e) \mid \phi \in Q_i^1 \}$ (here we use (2.1) to regard the elements of Q_i as maps $E_x \rightarrow H^\mathbb{C}$). Note that the choice of Q_i is equivalent to choosing an orbit of the action of P_{α_i} on E_x .

Note that if α_i lies $\sqrt{-1}\mathcal{A}'$ (in particular in $\sqrt{-1}\mathcal{A}$) then, as mentioned above $P_{\alpha_i}^1$ is trivial and a parabolic structure is simply the choice of a parabolic subgroup $Q_i \subset E(H^\mathbb{C})_{x_i}$ isomorphic to P_{α_i} (which is the more familiar concept).

Let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a collection of elements in $\sqrt{-1}\bar{\mathcal{A}}$. A **parabolic principal bundle over (X, D) of weight α** is a (holomorphic) principal bundle with a choice, for any i , of a parabolic structure of weight α_i on x_i . We will usually not specify in the notation the parabolic structure, so we will refer to them by the same symbol denoting the underlying principal bundle. Similarly we will often avoid referring to the weight α .

Naturally associated to the parabolic principal bundle E is the sheaf $PE(H^\mathbb{C})$ of **parabolic gauge transformations**, whose sections on $X \setminus D$ are the holomorphic sections g of $E(H^\mathbb{C})$, and near a marked point x_i , the sections are of the form $g(z) \exp(n/z)$ in some trivialization near x_i , such that $n \in \mathfrak{q}_i^1$ and g is holomorphic near x_i with $g(0) \in Q_i$ (this is a group because Q_i^1 is abelian and $gQ_i^1g^{-1} \subset Q_i^1$ for any $g \in Q_i$).

Again this simplifies if $\alpha_i \in \sqrt{-1}\mathcal{A}'$, in which case this is defined by holomorphic sections of $E(H^\mathbb{C})$ such that $g(x_i) \in Q_i$.

Of course we have the Lie algebra version $PE(\mathfrak{h}^\mathbb{C})$, whose sections are the sections u of $E(\mathfrak{h}^\mathbb{C})$ such that u is meromorphic with simple pole at x_i , $\text{Res}_{x_i} u \in \mathfrak{q}_i^1$ and the constant term $u(x_i) \in \mathfrak{q}_i$.

A useful geometric interpretation of parabolic gauge transformations is the following: near a parabolic point x_i , choose a trivialization of E , so that holomorphic sections of $E(H^\mathbb{C})$ (resp. $E(\mathfrak{h}^\mathbb{C})$) are identified with maps $\Delta \rightarrow H^\mathbb{C}$ (resp. $\mathfrak{h}^\mathbb{C}$), where Δ is the unit disc. Then one can check immediately:

$$(2.2) \quad \begin{aligned} PE(\mathfrak{h}^\mathbb{C}) &= \{ u : \Delta \rightarrow \mathfrak{h}^\mathbb{C}, \text{Ad}(|z|^{-\alpha_i})u(z) \text{ is bounded on } \Delta \setminus \{0\} \}, \\ PE(H^\mathbb{C}) &= \{ g : \Delta \rightarrow H^\mathbb{C}, |z|^{-\alpha_i}g(z)|z|^{\alpha_i} \text{ is bounded on } \Delta \setminus \{0\} \}. \end{aligned}$$

This means that parabolic transformations are holomorphic transformations on the punctured disc, which remain bounded with respect to the metric $|z|^{-2\alpha_i}$.

Remark 2.1. Note that in the case $\alpha_i \in \sqrt{-1}(\bar{\mathcal{A}} \setminus \mathcal{A}')$, a parabolic gauge transformation can be meromorphic, so can be considered as transforming the bundle E into another principal bundle which need not be isomorphic to E . This means that in this case, the holomorphic bundle underlying a parabolic bundle is only defined up to such transformations (this can be interpreted in terms of ‘parahoric structures’, see below). Nevertheless the sheaves $PE(H^\mathbb{C})$ and $PE(\mathfrak{h}^\mathbb{C})$ remain unchanged.

Remark 2.2. The definition of parabolic bundle can be extended to arbitrary α ’s, i.e., not necessarily in $\sqrt{-1}\bar{\mathcal{A}}$. However, this leads to more complicated objects that can be interpreted in terms of parahoric subgroups (see for example [11]) for which our analysis to prove the Hitchin-Kobayashi correspondence does not apply directly. But we do not need to go to these objects since by taking $\alpha \in \sqrt{-1}\bar{\mathcal{A}}$ we are able to parametrize all conjugacy classes of H and also of a non-compact real reductive group G with maximal compact H (see Appendix A) and hence obtain all possible monodromies at the punctures.

Example 2.3. The basic example is $H = U_n$ and $H^\mathbb{C} = GL_n\mathbb{C}$, so an $H^\mathbb{C}$ -bundle is in one-to-one correspondence with the holomorphic vector bundle $V = E(\mathbb{C}^n)$ associated to the fundamental representation of $GL_n\mathbb{C}$. The parabolic structure at the point x_i is given by a flag $V_{x_i} = V_i^1 \supset V_i^2 \supset \dots \supset V_i^{\ell_i}$, with corresponding weights $1 > \alpha_i^1 > \alpha_i^2 > \dots > \alpha_i^{\ell_i} \geq 0$. The matrix α_i is diagonal, with eigenvalues the α_i^j with multiplicity $\dim(V_i^j/V_i^{j+1})$, and the eigenvalues $\alpha_i^j - \alpha_i^k$ of $\text{ad}(\alpha_i)$ belong to $(-1, 1)$. The parabolic subalgebra consists of the endomorphisms of V_{x_i} , preserving the flag: it is the sum of the non positive eigenspaces of $\text{ad} \alpha_i$.

Remark 2.4. It is important here to note that our convention for the parabolic weights is different from the usual convention in the literature, since we take a decreasing sequence of weights. This is somehow a more natural choice in the group theoretic context: in Kähler quotients, it is natural to have an action of the group on the symmetric space on the right; then, if α_i lies in a positive Weyl chamber, it defines a point on the boundary at infinity of the symmetric space $H \backslash H^\mathbb{C}$, whose stabilizer is the parabolic subgroup whose Lie algebra is exactly the one defined above. See appendix D for details.

This has the consequence that the monodromy corresponding to the parabolic structure is $\exp(2\pi\sqrt{-1}\alpha_i)$ instead of $\exp(-2\pi\sqrt{-1}\alpha_i)$. Also note later the change of sign in the definition of the parabolic degree.

2.2. Hecke transformations and local coverings. We give now an interpretation of a parabolic structure for a weight α lying on a “bad” wall, i.e. such that $\alpha \in \sqrt{-1}(\bar{\mathcal{A}} \setminus \mathcal{A}')$ (see Appendix A). To do this, we consider the co-character $\chi : \mathbb{C}^* \rightarrow H^\mathbb{C}$ defined by an element $\lambda \in \sqrt{-1}\Lambda_{\text{cochar}}$ (see Appendix A) as $\chi(z) = \exp(\lambda \log z)$. This is meromorphic at the origin. We now choose a trivialization around a parabolic point $x \in D$ over the unit disc Δ . A trivialization of the principal bundle E around x is given by a section s of $E|_\Delta \rightarrow \Delta$. Let $g := \chi|_{\Delta \setminus \{0\}}$. This defines a holomorphic map

$$g : \Delta \setminus \{0\} \rightarrow H^\mathbb{C} \simeq E(H^\mathbb{C})_x.$$

We define the trivialization $s' := sg$, which is only defined on $\Delta \setminus \{0\}$. We then define a new principal $H^\mathbb{C}$ -bundle E' by setting that $E'|_{\Delta \setminus \{0\}} = E|_{\Delta \setminus \{0\}}$ and $E'|_\Delta$ is trivial, i.e. $E'|_\Delta = \Delta \times H^\mathbb{C}$, with the trivialization given by s' outside 0.

Let $\alpha \in \sqrt{-1}\mathfrak{t}$ we define the **Hecke transform** at the point x of (E, α, s) as $(E', \alpha' := \alpha + \lambda, s')$.

Note that the topological type of E' is generally different to that of E .

Since the Hecke transform $\alpha \mapsto \alpha'$ does not preserve $\bar{\mathcal{A}}$, the Hecke transform of a parabolic structure as we have defined it is not necessarily a parabolic structure. However we have the following.

Let $w \in W = N(T)/T$ be an element of the Weyl group. Let E be a holomorphic $H^\mathbb{C}$ -bundle, and let $\alpha \in \sqrt{-1}\mathfrak{t}$. By a Weyl transformation of (E, α, s) , we mean a triple $(E, w(\alpha), sg_w)$, where $g_w \in N(T) \subset H^\mathbb{C}$.

Proposition 2.5. *Let E be a parabolic bundle over X and let $x \in D$ be a parabolic point with weight $\alpha \in \sqrt{-1}(\bar{\mathcal{A}} \setminus \mathcal{A}')$. Then:*

(i) *there is a local finite cover p , ramified over the point x , such that after applying a Hecke transformation, together with a Weyl transformation to $E' = p^*E$, we obtain a weight $\alpha' \in \sqrt{-1}\mathcal{A}'$ at the point y mapping to x . In particular, this defines a parabolic structure on E' at y with weight α' ;*

(ii) *with the parabolic structure on E' defined as above, $PE(H^\mathbb{C}) = PE'(H^\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}}$, where $\mathbb{Z}/n\mathbb{Z}$ is the Galois group of the cover.*

Proof. Statement (i) is a consequence of (ii) in Proposition A.2. To prove statement (ii), we use (2.2): in the trivialization s , a gauge transformation g is a section of $PE(H^\mathbb{C})$ if $|z|^{-\alpha}g(z)|z|^\alpha$ is bounded. On the ramified cover $z = u^n$, we obtain for the pulled-back transformation that $|u|^{-n\alpha}g(u)|u|^{n\alpha}$ is bounded; the Hecke transformation followed by the Weyl transformation conjugates g into a gauge transformation $\tilde{g}(u) = g_w u^\lambda g(u) u^{-\lambda} g_w^{-1}$, so that

$$|u|^{-n\alpha-\lambda} g_w^{-1} \tilde{g}(u) g_w |u|^{n\alpha+\lambda} = g_w^{-1} |u|^{-nw(\alpha)-\lambda} \tilde{g}(u) |u|^{nw(\alpha)+\lambda} g_w$$

is bounded, that is

$$|u|^{-nw(\alpha)-\lambda} \tilde{g}(u) |u|^{nw(\alpha)+\lambda} \text{ is bounded,}$$

which by (2.2) is equivalent to saying that $\tilde{g} \in PE'(H^\mathbb{C})$. \square

Remark 2.6. Note that $PE'(H^\mathbb{C})$ is defined in terms of holomorphic sections of $E'(H^\mathbb{C})$ and is the more familiar sheaf of parabolic gauge transformations. So, after Hecke and Weyl transformations a parabolic structure with weight $\alpha \in \bar{\mathcal{A}} \setminus \mathcal{A}'$ is some sort of orbifold version of the familiar parabolic structure corresponding to a weight in $\sqrt{-1}\mathcal{A}'$.

2.3. Parabolic degree of parabolic reductions. Let E be a parabolic principal bundle over (X, D) of weight α and let $Q_i \subset E(H^\mathbb{C})_{x_i}$ denote the parabolic subgroups specified by the parabolic structure. For any standard parabolic subgroup $P \subset H^\mathbb{C}$, any antidominant character χ of \mathfrak{p} (see Appendix D), and any holomorphic reduction σ of the structure group of E from $H^\mathbb{C}$ to P we are going to define a number $\text{pardeg}(E)(\sigma, \chi) \in \mathbb{R}$, which we call the parabolic degree. This number will be the sum of two terms, one global and independent of the parabolic structure, and the other local and depending on the parabolic structure.

Before defining the parabolic degree, let us recall that the set of holomorphic reductions of the structure group of E from $H^\mathbb{C}$ to P is in one-to-one correspondence with the set of holomorphic sections σ of $E(H^\mathbb{C}/P)$ (the latter is the bundle associated to the action of $H^\mathbb{C}$ on the left on $H^\mathbb{C}/P$). Indeed, there is a canonical identification $E(H^\mathbb{C}/P) \simeq E/P$ and the quotient $E \rightarrow E/P$ has the structure of a P -principal bundle. So given a section σ of $E(H^\mathbb{C}/P)$ the pullback $E_\sigma := \sigma^*E$ is a P -principal bundle over X , and we can identify canonically $E \simeq E_\sigma \times_P H^\mathbb{C}$ as principal $H^\mathbb{C}$ -bundles. Equivalently, we can look at E_σ as a holomorphic subvariety $E_\sigma \subset E$ invariant under the action of $P \subset H^\mathbb{C}$ and inheriting a structure of principal bundle.

We need to show that a holomorphic reduction of the structure group of E to P is well-defined under parabolic gauge transformations, i.e. elements in $PE(H^\mathbb{C})$. Of course this is clear if the gauge transformation is holomorphic, but it can be meromorphic, of the form $g(z) = \exp(n/z)$ at a parabolic point with $\alpha \in \sqrt{-1}(\bar{A} \setminus A')$. Nevertheless after gauge transforming the bundle we still have a holomorphic reduction at the marked point, as follows from the following lemma:

Lemma 2.7. *Let Δ be the unit disc in \mathbb{C} . Let $P \subset H^\mathbb{C}$ be a parabolic subgroup. Let $\sigma : \Delta \rightarrow H^\mathbb{C}/P$ be a holomorphic map and let $g : \Delta \setminus \{0\} \rightarrow H^\mathbb{C}$ be a meromorphic map at 0. Then the map $g \cdot \sigma : \Delta \setminus \{0\} \rightarrow H^\mathbb{C}/P$ defined by $g(z) \cdot \sigma(z)$ can be extended to a holomorphic map at $0 \in \Delta$.*

Proof. We can embed $H^\mathbb{C}/P$ in \mathbb{P}^N with $H^\mathbb{C} \subset \mathrm{SL}(N+1, \mathbb{C})$. The case then reduces to $\mathrm{SL}(N+1, \mathbb{C})$ acting on \mathbb{P}^N . The result is then obvious. \square

Now fix a standard parabolic subgroup $P = P_A \subset H^\mathbb{C}$, an antidominant character χ of P , and a holomorphic reduction σ of the structure group of E from $H^\mathbb{C}$ to P .

The global term in $\mathrm{pardeg}(E)(\sigma, \chi)$ is the degree $\deg(E)(\sigma, \chi)$ defined in [20, (A.57)]. We introduce it here using Chern-Weil theory instead of the algebraic constructions of [op. cit.].

Let E_σ be the P -principal bundle corresponding to the reduction σ . Given a character $\chi : \mathfrak{p}_A \rightarrow \mathbb{C}$, the degree is defined by the Chern-Weil formula

$$(2.3) \quad \deg(E)(\sigma, \chi) := \frac{\sqrt{-1}}{2\pi} \int_X \chi_*(F_A)$$

for any P -connection A on E_σ . Since $P \cap H$ is a maximal compact subgroup of P and the inclusion $P \cap H \hookrightarrow P$ is a homotopy equivalence (see Appendix D), one can evaluate (2.3) using a $P \cap H$ -connection, and it follows that $\deg(E)(\sigma, \chi)$ is a real number.

Proposition 2.8. *The degree $\deg(E)(\sigma, \chi)$ is well-defined under transformations in $PE(H^\mathbb{C})$.*

Proof. The only transformations on the punctured disk which do not extend at the origin are of the type $\exp n/z$, so they are homotopic to the identity on the punctured disk. It follows immediately that the degree remains unchanged. \square

At each marked point x_i we have two parabolic subgroups of $E(H^\mathbb{C})_{x_i}$ equipped with an antidominant character:

- one coming from the parabolic structure, (Q_i, α_i) , where α_i can be considered as a strictly antidominant character of Q_i ;
- one coming from the reduction, $(E_\sigma(P)_{x_i}, \chi)$.

In appendix D, we define a relative degree $\deg((Q_i, \alpha_i), (E_\sigma(P)_{x_i}, \chi))$ of such a pair. Then we define the parabolic degree as follows:

$$(2.4) \quad \mathrm{pardeg}_\alpha(E)(\sigma, \chi) := \deg(E)(\sigma, \chi) - \sum_i \deg((Q_i, \alpha_i), (E_\sigma(P)_{x_i}, \chi)).$$

When it is clear from the context we will omit the subscript α in the notation of the parabolic degree.

The definition of parabolic degree of a reduction also makes sense if one takes as parabolic subgroup the whole $H^\mathbb{C}$. In this case the set of antidominant characters is simply $\mathrm{Hom}_\mathbb{R}(\mathfrak{z}, \sqrt{-1}\mathbb{R})$, where \mathfrak{z} is the center of \mathfrak{h} . Trivially, in this case there is a unique reduction of the structure group, which we denote by σ_0 . We then define for any $\chi \in \mathrm{Hom}_\mathbb{R}(\mathfrak{z}, \sqrt{-1}\mathbb{R})$

$$\mathrm{pardeg}_\chi E := \mathrm{pardeg}_\alpha(E)(\sigma_0, \chi).$$

2.4. Analytic formula for $\deg(E)(\sigma, \chi)$. Let $s \in \sqrt{-1}\mathfrak{h}$ be any element, let $P = P_s \subset H^\mathbb{C}$ be the corresponding parabolic subgroup and let $\chi : \mathfrak{p} \rightarrow \mathbb{C}$ be the antidominant character defined as $\chi(\alpha) = \langle \alpha, s \rangle$, where $\langle \cdot, \cdot \rangle : \mathfrak{h}^\mathbb{C} \times \mathfrak{h}^\mathbb{C} \rightarrow \mathbb{C}$ is the extension of the invariant pairing on \mathfrak{h} to a Hermitian pairing. Note that the intersection of P and H can be identified with the centralizer of s in H :

$$(2.5) \quad P \cap H = Z_H(s) = \{h \in H \mid \text{Ad}(h)(s) = s\}.$$

Let σ be a holomorphic reduction of the structure group of E from $H^\mathbb{C}$ to P , and E_σ the corresponding P -principal bundle. Let $N \subset P$ be the unipotent part of P , and choose the Levi subgroup $L = Z_{H^\mathbb{C}}(s) \subset P$. Then there is a well-defined L -action on E_σ/N which turns it into a principal L -bundle, which we denote $E_{\sigma,L}$. (In the vector bundle case, the P -reduction is a flag of sub-vector bundles, and the L -bundle is the associated graded bundle). Note that since $L = Z_{H^\mathbb{C}}(s)$, the element $s \in \sqrt{-1}\mathfrak{h}$ defines a canonical section

$$(2.6) \quad s_\sigma \in \Gamma(E_{\sigma,L}(\mathfrak{l} \cap \sqrt{-1}\mathfrak{h})).$$

Let h be a smooth metric on E , defined on the whole curve X , and let \mathbf{E} be the H -principal bundle obtained by reducing the structure group of E from $H^\mathbb{C}$ to H using h . Combining σ and h we obtain a reduction of the structure group of E from $H^\mathbb{C}$ to $P \cap H$. Denote by \mathbf{E}_σ the resulting bundle. But $P \cap H$ is a compact form of L , and the complexified bundle clearly identifies to $E_{\sigma,L}$, so we can also think of \mathbf{E}_σ as the bundle $E_{\sigma,L}$ equipped with the induced metric. From this point of view, the section s_σ above can be seen as a section

$$(2.7) \quad s_{\sigma,h} \in \Gamma(\mathbf{E}_\sigma(\sqrt{-1}\mathfrak{h})) \simeq \Gamma(\mathbf{E}(\sqrt{-1}\mathfrak{h})).$$

Note the difference between (2.6) and (2.7): the section s_σ is canonical, while different choices of h lead to different sections $s_{\sigma,h}$ of $E(\mathfrak{h}^\mathbb{C})$.

We now introduce some notation. Let V be a Hermitian vector space, let $\rho : \mathfrak{h} \rightarrow \mathfrak{u}(V)$ be a morphism of Lie algebras, and denote also by $\rho : \mathfrak{h}^\mathbb{C} \rightarrow \text{End } V$ its complex extension. Choose elements $a \in \sqrt{-1}\mathfrak{h}$ and $v \in V$. Then $\rho(a)$ diagonalizes and has real eigenvalues, so that we may write $v = \sum v_j$ in such a way that $\rho(a)(v) = \sum l_j v_j$. Now, for any function $f : \mathbb{R} \rightarrow \mathbb{R}$ we define $f(a)(v) := \sum f(l_j) v_j$.

Lemma 2.9. *Define the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ as $\phi(0) = 0$ and as $\phi(x) = x^{-1}$ if $x \neq 0$. Applying the previous definition to the adjoint representation, and extending it to sections of $E(\mathfrak{h}^\mathbb{C}) \otimes K$, we have:*

$$\deg(E)(\sigma, \chi) = \frac{\sqrt{-1}}{2\pi} \left(\int_X \langle F_h, s_{\sigma,h} \rangle + \int_X \langle \phi(s_{\sigma,h})(\bar{\partial}s_{\sigma,h}), \bar{\partial}s_{\sigma,h} \rangle \right).$$

The proof follows from identifying the RHS of the formula with the curvature of the Chern connection of $E_{\sigma,L}$ for the metric h . More precisely,

Lemma 2.10. $\langle F_{h,L}, s_\sigma \rangle = \langle F_h, s_{\sigma,h} \rangle + \langle \phi(s_{\sigma,h})(\bar{\partial}s_{\sigma,h}), \bar{\partial}s_{\sigma,h} \rangle$.

Proof. Easy computation. □

Our aim in the remainder of this section is to state and prove an analogue of Lemma 2.9 giving the parabolic degree. For that it will be necessary to replace the metric h (which was chosen to be smooth on the whole X) by a metric which blows up at the divisor D at a speed specified by the parabolic weights α_i .

2.5. α -adapted metrics and parabolic degree. It may be useful here to remind in a few words how we write local formulas for the metrics of principal bundles. There is a right action $h \mapsto h \cdot g$ of gauge transformations $g \in \Gamma(E(H^\mathbb{C}))$ on metrics $h \in \Gamma(E/H)$, which identifies in each fibre to the standard action of $H^\mathbb{C}$ on the symmetric space $H \backslash H^\mathbb{C}$. Let τ be the involution of $H^\mathbb{C}$ fixing H . A local

trivialization e of E defines a metric h_0 ; another metric is given by $h = h_0 \cdot g$ for some g with values in $H^\mathbb{C}$; of course h depends only on $\tau(g)^{-1}g$ and we identify $h = \tau(g)^{-1}g$ (in the linear case, this is just writing $h = g^*g$). Of course it is always possible to move g by an element of H so that $\tau(g)^{-1} = g$ and $h = g^2$, which will be our most usual way to write metrics explicitly.

Let now consider a parabolic bundle E , and a metric $h \in \Gamma(X \setminus D; E/H)$ defined away from the divisor D .

Definition 2.11. *We say that h is an α -adapted metric if for any parabolic point x_i the following holds. Let $e_i \in E_{x_i}$ be an element belonging to the P_{α_i} orbit specified by the parabolic structure. Choose a local holomorphic coordinate z , and extend the trivialization e_i into a holomorphic trivialization of E around x_i . Then we can write, near x_i ,*

$$(2.8) \quad h = (|z|^{-\alpha_i} e^c)^2,$$

with $\text{Ad}(|z|^{-\alpha_i} c) = o(\log |z|)$, $\text{Ad}(|z|^{-\alpha_i} dc) \in L^2$ and $\text{Ad}(|z|^{-\alpha_i} F_h) \in L^1$.

(Giving bounds on $\text{Ad}(|z|^{-\alpha_i})f$ amounts to giving bounds on f with respect to the metric $|z|^{-2\alpha_i}$). Note that the condition is well defined: if we change the trivialization by a gauge transformation $g \in \Gamma(X, PE(H^\mathbb{C}))$, the resulting metric will remain at bounded distance of h_0 in $H \setminus H^\mathbb{C}$; also, the L^2 and L^1 conditions on dc and F_h are conformally invariant.

We are now ready to state and prove the analogue of Lemma 2.9 for the parabolic degree.

Lemma 2.12. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be defined as in Lemma 2.9, and let h be an α -adapted metric. Then:*

$$\text{pardeg}_\alpha(E)(\sigma, \chi) = \frac{\sqrt{-1}}{2\pi} \left(\int_{X \setminus D} \langle F_h, s_{\sigma, h} \rangle + \langle \phi(s_{\sigma, h})(\bar{\partial}s_{\sigma, h}), \bar{\partial}s_{\sigma, h} \rangle \right).$$

Proof. For any $v > 0$ let $X_v = \{x \in X \mid d(x, D) \geq e^{-v}\}$ and $B_v = X \setminus X_v$. Let h_v be a smooth metric on E (defined on the whole X) which coincides with h in a neighborhood of $X_v \subset X$. The metrics h and h_v induce metrics h_L and $h_{L,v}$ on $E_{\sigma, L}$, and we denote by $\mathbf{E}_{\sigma, v}$ the resulting Hermitian holomorphic bundle. Denote the curvatures of h_L and $h_{L,v}$ by $F_{h, L}$ and $F_{h_v, L}$. It follows from the definition of $\deg(E)(\sigma, \chi)$ that

$$\begin{aligned} \deg(E)(\sigma, \chi) &= \frac{\sqrt{-1}}{2\pi} \int_X \langle F_{h_v, L}, s_\sigma \rangle \\ &= \frac{\sqrt{-1}}{2\pi} \left(\int_{X_v} \langle F_{h_v, L}, s_\sigma \rangle + \int_{B_v} \langle F_{h_v, L}, s_\sigma \rangle \right). \end{aligned}$$

By Lemma 2.10 we have

$$\begin{aligned} \lim_{v \rightarrow \infty} \int_{X_v} \langle F_{h_v, L}, s_\sigma \rangle &= \int_X \langle F_{h, L}, s_\sigma \rangle \\ &= \int_{X \setminus D} \langle F_h, s_{\sigma, h} \rangle + \langle \phi(s_{\sigma, h})(\bar{\partial}s_{\sigma, h}), \bar{\partial}s_{\sigma, h} \rangle. \end{aligned}$$

Hence we need to prove that the remaining integral converges to the local terms in the definition of the parabolic degree, i.e.:

$$(2.9) \quad \lim_{v \rightarrow \infty} \frac{\sqrt{-1}}{2\pi} \int_{B_v} \langle F_{h_v, L}, s_\sigma \rangle = \sum_i \deg((P, \chi), (Q_i, \alpha_i)).$$

Observe that on a small ball, we can use a holomorphic trivialization of the bundle E_σ , and the connection form becomes $A_v = h_{L,v}^{-1} \partial h_{L,v}$ with curvature $F = dA_v$. It

follows that the quantity we have to study is

$$\lim_{v \rightarrow \infty} \frac{\sqrt{-1}}{2\pi} \int_{\partial B_v} \langle A_v, s \rangle = \lim_{v \rightarrow \infty} \frac{\sqrt{-1}}{2\pi} \int_{\partial B_v} \langle h_L^{-1} \partial h_L, s \rangle,$$

since $h_{L,v}$ coincides with h_L in a neighbourhood of X_v .

We can choose a holomorphic trivialization of E in which the reduction σ has constant coefficients. This induces also a holomorphic trivialization of the L -bundle $E_{\sigma,L}$. We write the metric h on E in this trivialization as $h = g^2$, where $g = |z|^{-\alpha_i} e^c$, and we decompose

$$g = k(z)l(z)n(z), \quad \text{with } k(z) \in H, l(z) \in L, n(z) \in N.$$

The pair $(k(z), l(z))$ is defined only up to the action of $P \cap H$, so we can as well suppose that $\tau(l) = l^{-1}$. It then follows that

$$h_L = l(z)^2.$$

Then, because s is L -invariant, the limit to calculate becomes

$$\lim_{v \rightarrow \infty} \frac{\sqrt{-1}}{\pi} \int_{\partial B_v} \langle \partial l l^{-1}, s \rangle.$$

Now one has

$$\partial l l^{-1} = \text{Ad}(k)^{-1}(\partial g g^{-1}) - k^{-1} \partial k - \text{Ad}(l)(\partial n n^{-1}).$$

Observe that since $s \in \sqrt{-1}\mathfrak{h}$ and $k \in H$, the term $\frac{\sqrt{-1}}{\pi} \int_{\partial B_v} \langle k^{-1} \partial k, s \rangle$ reduces to $\frac{1}{2\pi} \int_{\partial B_v} \langle \sqrt{-1} k^{-1} \partial_\theta k, s \rangle$ which vanishes because k extends in the ball B_v . Therefore the limit reduces to

$$(2.10) \quad \lim_{v \rightarrow \infty} \frac{\sqrt{-1}}{\pi} \int_{\partial B_v} \langle \text{Ad}(k)^{-1}(\partial g g^{-1}), s \rangle.$$

On the other hand, decompose similarly $e^{t\alpha} = \tilde{k}(t)\tilde{p}(t)$ with $\tilde{k}(t) \in H$ and $\tilde{p}(t) \in P$ then $\langle s \cdot e^{-t\alpha}, \alpha \rangle = \langle \text{Ad}(\tilde{k}(t))s, \alpha \rangle$, so we obtain:

$$(2.11) \quad \deg((P, \chi), (Q_i, \alpha_i)) = \lim_{t \rightarrow \infty} \langle s \cdot e^{-t\alpha}, \alpha \rangle = \lim_{t \rightarrow \infty} \langle s, \text{Ad}(\tilde{k}(t))^{-1} \alpha \rangle.$$

If we have exactly the model behavior $g = |z|^{-\alpha_i}$, then one has $k(z) = \tilde{k}(-\ln|z|)$ and $\partial g g^{-1} = -\frac{\alpha_i}{2} \frac{dz}{z}$, so

$$\frac{\sqrt{-1}}{\pi} \int_{\partial B_v} \langle \text{Ad}(k(z))^{-1}(\partial g g^{-1}), s \rangle = \langle \text{Ad}(\tilde{k}(v))^{-1} \alpha_i, s \rangle,$$

so the two limits (2.10) and (2.11) are the same. It is then easy to check that this remains true if $g = |z|^{-\alpha_i} e^c$, where c is a perturbation satisfying the conditions below (2.8). \square

3. PARABOLIC G -HIGGS BUNDLES

3.1. Definition of parabolic G -Higgs bundle. Following the definition and notation in Appendix B, let $G = (G, H, \theta, B)$ be a real reductive Lie group. Let X be a compact connected Riemann surface and let $\{x_1, \dots, x_r\}$ be a finite set of different points of X . Let $D = x_1 + \dots + x_r$ be the corresponding effective divisor. Let E be a parabolic principal $H^\mathbb{C}$ -bundle over (X, D) . Let $E(\mathfrak{m}^\mathbb{C})$ be the bundle associated to E via the isotropy representation (see Appendix B).

We now define the **sheaf $PE(\mathfrak{m}^\mathbb{C})$ of parabolic sections of $E(\mathfrak{m}^\mathbb{C})$** and the **sheaf $NE(\mathfrak{m}^\mathbb{C})$ of strictly parabolic sections of $E(\mathfrak{m}^\mathbb{C})$** . These consist of meromorphic sections of $E(\mathfrak{m}^\mathbb{C})$, holomorphic on $X \setminus D$, with singularities of a certain type on D . More precisely, choose a trivialization $e \in E$ near x_i , such that at

the point x_i the parabolic weight is $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$. In the trivialization e , we can decompose the bundle $E(\mathfrak{m}^{\mathbb{C}})$ under the eigenvalues of $\text{ad}(\alpha_i)$ (acting on $\mathfrak{m}^{\mathbb{C}}$),

$$E(\mathfrak{m}^{\mathbb{C}}) = \oplus_{\mu} \mathfrak{m}_{\mu}^{\mathbb{C}}.$$

Decompose accordingly a section φ of $E(\mathfrak{m}^{\mathbb{C}})$ as $\varphi = \sum \varphi_{\mu}$, then we say that φ is a section of the sheaf $PE(\mathfrak{m}^{\mathbb{C}})$ (resp. $NE(\mathfrak{m}^{\mathbb{C}})$) if φ is meromorphic at x_i , and φ_{μ} has order

$$(3.1) \quad v(\varphi_{\mu}) \geq -\lfloor -\mu \rfloor \quad \text{resp.} \quad v(\varphi_{\mu}) > -\lfloor -\mu \rfloor.$$

This means that if $a-1 < \mu \leq a$ (resp. $a-1 \leq \mu < a$) for some integer a , then $\varphi_{\mu} = O(z^a)$.

An equivalent way to define it is to say that a section of $PE(\mathfrak{m}^{\mathbb{C}})$ is a holomorphic section of the bundle

$$\oplus_{\mu} \mathfrak{m}_{\mu}^{\mathbb{C}}(\lfloor -\mu \rfloor x_i).$$

Of course, in general, if we take a holomorphic bundle with some decomposition at a point, this construction does not make sense, because the result depends on the extension of the decomposition near the point. However, the following lemma proves that, in our case, the definition does not depend on the choice of the trivialization.

Lemma 3.1. *The action of a section g of the sheaf $PE(H^{\mathbb{C}})$ preserves the set of sections of the sheaves $PE(\mathfrak{m}^{\mathbb{C}})$ and $NE(\mathfrak{m}^{\mathbb{C}})$.*

Proof. We write the proof only for $PE(\mathfrak{m}^{\mathbb{C}})$. The first case is that of a section $g = \exp(n/z)$ with $n \in \mathfrak{q}_i^1$. Then, for $\varphi \in E(\mathfrak{m}^{\mathbb{C}})$,

$$\begin{aligned} \text{Ad}(g)\varphi &= e^{\text{ad} \frac{n}{z}} \varphi \\ &= \varphi + \left[\frac{n}{z}, \varphi\right] + \frac{1}{2} \left[\frac{n}{z}, \left[\frac{n}{z}, \varphi\right]\right] + \cdots \end{aligned}$$

Since $[\alpha_i, n] = -n$, one has $[n, \mathfrak{m}_{\mu}^{\mathbb{C}}] \subset \mathfrak{m}_{\mu-1}^{\mathbb{C}}$, so $\text{Ad}(g)\varphi$ satisfies (3.1) if φ does.

The second case is that of a constant $g \in Q_i$: it is clear that nothing is changed if g belongs to the Levi subgroup L_{α_i} (see §D.1), so we can suppose that g belongs to the unipotent part of Q_i : let us write $g = \exp(n)$ with

$$(3.2) \quad n \in \oplus_{\lambda < 0} \mathfrak{h}_{\lambda}^{\mathbb{C}},$$

where $\mathfrak{h}^{\mathbb{C}} = \oplus_{\lambda} \mathfrak{h}_{\lambda}^{\mathbb{C}}$ is the eigenspace decomposition of $\mathfrak{h}^{\mathbb{C}}$ under the action of $\text{ad}(\alpha_i)$. Then, as above,

$$\text{Ad}(g)\varphi = \varphi + [n, \varphi] + \frac{1}{2} [n, [n, \varphi]] + \cdots$$

Because of (3.2), if $\varphi \in \mathfrak{m}_{\mu}^{\mathbb{C}}$ then $[n, \varphi] \in \oplus_{\lambda < 0} \mathfrak{m}_{\lambda+\mu}^{\mathbb{C}}$, and more generally $\text{Ad}(g)\varphi - \varphi \in \oplus_{\mu' < \mu} \mathfrak{m}_{\mu'}^{\mathbb{C}}$, so that $\text{Ad}(g)\varphi$ again satisfies (3.1).

The third and last case consists in applying a holomorphic change of trivialization by a $g \in H^{\mathbb{C}}$ such that $g(x_i) = 1$. Let us write $g = \exp(zu)$ with $u \in \mathfrak{h}^{\mathbb{C}}$ holomorphic, and decompose $u = \oplus_{\lambda} u_{\lambda}$. Then, again,

$$\text{Ad}(g)\varphi = \varphi + z[u, \varphi] + \frac{z^2}{2} [u, [u, \varphi]] + \cdots$$

Here the important point is that all the eigenvalues λ satisfy $|\lambda| \leq 1$, so that if $\varphi \in \mathfrak{m}_{\mu}^{\mathbb{C}}$, then $\text{ad}(u)^k \varphi \in \oplus_{\mu' \leq \mu+k} \mathfrak{m}_{\mu'}^{\mathbb{C}}$ and $z^k \text{ad}(u)^k \varphi$ again satisfies (3.1). This concludes the proof that the sheaf $PE(\mathfrak{m}^{\mathbb{C}})$ is well defined. \square

The sheaves $PE(\mathfrak{m}^{\mathbb{C}})$ and $NE(\mathfrak{m}^{\mathbb{C}})$ have a much simpler description when $\alpha_i \in \sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$, where $\mathcal{A}'_{\mathfrak{g}}$ is the space of $\alpha \in \bar{\mathcal{A}}$ such that the eigenvalues of $\text{ad} \alpha$ have modulus smaller than 1, not only on \mathfrak{h} , but on the whole \mathfrak{g} , and in particular on \mathfrak{m}

(one can often choose \mathcal{A} so that this happens). To show this, consider for $\alpha \in \sqrt{-1}\mathfrak{h}$ the subspaces of $\mathfrak{m}^{\mathbb{C}}$ defined by

$$\begin{aligned}\mathfrak{m}_{\alpha} &= \{v \in \mathfrak{m}^{\mathbb{C}} : \text{Ad}(e^{t\alpha})v \text{ is bounded as } t \rightarrow \infty\} \\ \mathfrak{m}_{\alpha}^0 &= \{v \in \mathfrak{m}^{\mathbb{C}} : \text{Ad}(e^{t\alpha})v = v \text{ for every } t\}.\end{aligned}$$

We have that $\mathfrak{m}_{\alpha}^0 \subset \mathfrak{m}_{\alpha}$ and we can choose a complement \mathfrak{n}_{α} so that $\mathfrak{m}_{\alpha} = \mathfrak{m}_{\alpha}^0 \oplus \mathfrak{n}_{\alpha}$.

Recall that when $\alpha_i \in \sqrt{-1}\mathcal{A}'$, the parabolic structure at x_i is given by a parabolic subgroup $Q_i \subset E(H^{\mathbb{C}})_{x_i}$ isomorphic to P_{α_i} . This determines an isomorphism of $E(\mathfrak{m}^{\mathbb{C}})_{x_i}$ with $\mathfrak{m}^{\mathbb{C}}$. We can then define the subspaces \mathfrak{m}_i , \mathfrak{m}_i^0 and \mathfrak{n}_i of $E(\mathfrak{m}^{\mathbb{C}})_{x_i}$ corresponding to \mathfrak{m}_{α_i} , $\mathfrak{m}_{\alpha_i}^0$ and \mathfrak{n}_{α_i} , respectively. Then, when $\alpha_i \in \sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$ the **sheaf $PE(\mathfrak{m}^{\mathbb{C}})$ of parabolic sections of $E(\mathfrak{m}^{\mathbb{C}})$** is the sheaf of local holomorphic sections ψ of $E(\mathfrak{m}^{\mathbb{C}})$ such that $\psi(x_i) \in \mathfrak{m}_i$. Similarly, the **sheaf $NE(\mathfrak{m}^{\mathbb{C}})$ of nilpotent sections of $E(\mathfrak{m}^{\mathbb{C}})$** is the sheaf of local holomorphic sections ψ of $E(\mathfrak{m}^{\mathbb{C}})$ such that $\psi(x_i) \in \mathfrak{n}_i$. We then have short exact sequences of sheaves

$$0 \rightarrow PE(\mathfrak{m}^{\mathbb{C}}) \rightarrow E(\mathfrak{m}^{\mathbb{C}}) \rightarrow \bigoplus_i E(\mathfrak{m}^{\mathbb{C}})_{x_i}/\mathfrak{m}_i \rightarrow 0,$$

and

$$0 \rightarrow NE(\mathfrak{m}^{\mathbb{C}}) \rightarrow E(\mathfrak{m}^{\mathbb{C}}) \rightarrow \bigoplus_i E(\mathfrak{m}^{\mathbb{C}})_{x_i}/\mathfrak{n}_i \rightarrow 0.$$

After these preliminaries, we can define a **parabolic G -Higgs bundle** to be a pair of the form (E, ϕ) , where E is a parabolic $H^{\mathbb{C}}$ -principal bundle over (X, D) and ϕ is a holomorphic section of $PE(\mathfrak{m}^{\mathbb{C}}) \otimes K_{\log D}$. We shall say that (E, ϕ) is **strictly parabolic** if in addition ϕ is a section of $NE(\mathfrak{m}^{\mathbb{C}}) \otimes K_{\log D}$. The section ϕ is the Higgs field. By $K_{\log D}$, as usual, we denote the line bundle $K(D)$.

We now define the residue of ϕ at the points x_i . This is again much simpler if the weights $\alpha_i \in \sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$. The Higgs field ϕ is then a meromorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$ with a simple pole at $x_i \in D$ and the **residue** of ϕ at x_i is hence an element

$$\text{Res}_{x_i} \phi \in \mathfrak{m}_i.$$

We denote the projection of $\text{Res}_{x_i} \phi$ in \mathfrak{m}_i^0 by $\text{Gr Res}_{x_i} \phi$. As we will see, this projection is what is relevant in relation to local systems as well as to define the appropriate moduli space.

More generally, if $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$, locally ϕ is a section of $PE(\mathfrak{m}^{\mathbb{C}}) \otimes K_{\log D}$, with $PE(\mathfrak{m}^{\mathbb{C}}) = \bigoplus_{\mu} \mathfrak{m}_{\mu}^{\mathbb{C}}(-\lfloor \mu \rfloor x_i)$. Choosing a trivialization of $\mathcal{O}(D)$ at the point x_i , we can identify the fibre of $PE(\mathfrak{m}^{\mathbb{C}})$ at x_i with $\mathfrak{m}^{\mathbb{C}}$, and then project the residue of ϕ at x_i on the centralizer of $\exp(\alpha_i)$ in $\mathfrak{m}^{\mathbb{C}}$. We denote this projection by $\text{Gr Res}_{x_i} \phi$. (In the case where $\alpha_i \in \sqrt{-1}\mathcal{A}'$, this is just the projection on the Levi part \mathfrak{m}_i^0 as mentioned above.)

In concrete terms, if we have a local coordinate z near x_i and a holomorphic trivialization of E near x_i , we can write

$$\phi = \sum_{\mu} \frac{\phi_{\mu}}{z^{\lfloor -\mu \rfloor}} \frac{dz}{z},$$

with ϕ_{μ} is holomorphic, and then

$$\text{Gr Res}_{x_i} \phi = \sum_{\mu \in \mathbb{Z}} \phi_{\mu}(0).$$

If we change the coordinate, $z' = fz$, then

$$\phi = \sum_{\mu} f^{\lfloor -\mu \rfloor} \frac{\phi_{\mu}}{(z')^{\lfloor -\mu \rfloor}} \frac{dz'}{z'},$$

and $\sum_{\mu \in \mathbb{Z}} \phi_\mu(0)$ is changed into

$$\sum_{\mu \in \mathbb{Z}} f(0)^{-\mu} \phi_\mu(0) = \text{Ad}(e^{-\alpha_i \ln f(0)}) \sum_{\mu \in \mathbb{Z}} \phi_\mu(0),$$

for any choice of logarithm of $f(0)$. So we deduce that $\text{Gr Res}_{x_i} \phi$ is well defined up to the action of the 1-parameter group generated by α_i . Note that this ambiguity exists only in the case where $\text{ad}(\alpha_i)$ has non zero integer eigenvalues on $\mathfrak{m}^\mathbb{C}$.

We must also verify that the definition of $\text{Gr Res}_{x_i} \phi$ does not depend on the choice of a gauge $g \in PE(H^\mathbb{C})$. The only significant case is that of a $g(z) = \exp \frac{n}{z}$ with $n \in \mathfrak{q}_i^1$. Then

$$\text{Ad}(g(z))\phi = \phi + \frac{1}{z}[n, \phi] + \frac{1}{2z^2}[n, [n, \phi]] + \dots$$

Since $[n, \mathfrak{m}_\mu^\mathbb{C}] \subset \mathfrak{m}_{\mu-1}^\mathbb{C}$, it follows that $\text{Gr Res}_{x_i} \phi$ is transformed into

$$\text{Ad}(e^n) \text{Gr Res}_{x_i} \phi.$$

The other cases are left to the reader.

Let us now see what the definition means in three simple cases. The first case is the one of Example 2.3, that is $G = GL_n \mathbb{C}$. Here a G -Higgs bundle is just an ordinary Higgs bundle. In that case, the ambiguity on $\text{Gr Res}_{x_i} \phi$ does not show up. The Higgs field ϕ is a meromorphic 1-form with simple poles at the x_i such that $\text{Res}_{x_i} \phi$ preserves the flag, and $\text{Gr Res}_{x_i} \phi$ is the endomorphism induced by $\text{Res}_{x_i} \phi$ on the associated graded space (hence our notation $\text{Gr Res}_{x_i} \phi$).

The second example, in which the ambiguity shows up, is $G = SL_2 \mathbb{R}$ and $H = U_1$. A G -Higgs bundle in this case is given by an $H^\mathbb{C}$ -bundle E (that is just a line bundle L), and a Higgs field ϕ which is 1-form with values in $E(\mathfrak{m}^\mathbb{C}) = L^2 \oplus L^{-2}$. The parabolic structure at a point x is given by a weight α which we can take in the interval $(-\frac{1}{2}, \frac{1}{2}]$. The eigenvalues of $\text{ad}(\alpha)$ on $\mathfrak{m}^\mathbb{C}$ are $\pm 2\alpha$, and integer eigenvalues ± 1 appear only for $\alpha = \frac{1}{2}$. Let us examine this case more carefully. If we represent $\alpha \in \sqrt{-1}\mathfrak{u}_1$ as the matrix

$$(3.3) \quad \alpha = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},$$

then we obtain, for holomorphic ϕ_\pm ,

$$(3.4) \quad \phi = \begin{pmatrix} 0 & z\phi_+ \\ \frac{1}{z}\phi_- & 0 \end{pmatrix} \frac{dz}{z}, \quad \text{Gr Res}_x \phi = \begin{pmatrix} 0 & \phi_+(0) \\ \phi_-(0) & 0 \end{pmatrix},$$

and $\text{Gr Res}_{x_i} \phi$ is defined up to

$$\begin{pmatrix} 0 & \phi_+(0) \\ \phi_-(0) & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & c\phi_+(0) \\ c^{-1}\phi_-(0) & 0 \end{pmatrix}.$$

Of course there is another way to consider the same object: we can think of (E, ϕ) as a $GL_2 \mathbb{C}$ -Higgs bundle, the underlying holomorphic bundle of which is $L \oplus L^{-1}$, with parabolic weights $(\frac{1}{2}, -\frac{1}{2})$. Actually, to get weights in a correct interval, it is better to consider $L \oplus L^{-1}(D)$, with weights $(\frac{1}{2}, \frac{1}{2})$. If (e_1, e_2) is a trivialization of $L \oplus L^{-1}$, then $(e_1, z^{-1}e_2)$ is a trivialization of $L \oplus L^{-1}(D)$ and the Higgs field (3.4) becomes

$$\phi = \begin{pmatrix} 0 & \phi_+ \\ \phi_- & 0 \end{pmatrix} \frac{dz}{z},$$

so that $\text{Gr Res}_{x_i} \phi$ coincides with the one obtained in (3.4), with the same ambiguity coming from the choice of a trivialization of $\mathcal{O}(D)$.

The third and last example is just our second example considered for the group $G = SL_2 \mathbb{C}$, so $H = SU_2$. Then $H^\mathbb{C} = SL_2 \mathbb{C}$ and $\mathfrak{m}^\mathbb{C} = \mathfrak{sl}_2 \mathbb{C}$. The difference is

now that the weight (3.3) has nonzero integer eigenvalues on $\mathfrak{h}^{\mathbb{C}}$ itself, so one must consider gauge transformations which are meromorphic:

$$g = \begin{pmatrix} O(1) & O(z) \\ O(\frac{1}{z}) & O(1) \end{pmatrix}.$$

This gauge transformation becomes holomorphic in the $GL_2\mathbb{C}$ -gauge $(e_1, z^{-1}e_2)$ considered above, and all the data of the G -Higgs bundle is equivalent to that of the $GL_2\mathbb{C}$ -Higgs bundle obtained after this Hecke transformation. Nevertheless, it is useful to have a general definition which does not require to change the group.

3.2. Stability of parabolic G -Higgs bundles. The notion of stability, semistability and polystability of a parabolic G -Higgs bundle depends on an element of $\sqrt{-1}\mathfrak{z}$, where \mathfrak{z} is the centre of \mathfrak{h} . We will develop the theory here for any element in $\sqrt{-1}\mathfrak{z}$. However, in order to relate parabolic G -Higgs bundles to G -local systems, one requires this element to lie also in the centre of \mathfrak{g} , and actually to be 0. This is always the case in particular if G is semisimple.

Let $s \in \sqrt{-1}\mathfrak{h}$. We consider the parabolic subgroup P_s of $H^{\mathbb{C}}$, and the corresponding Levi subgroup L_s as defined in §D.1. Let χ_s be the corresponding antidominant character of \mathfrak{p}_s , where \mathfrak{p}_s is the Lie algebra of P_s . We consider

$$\begin{aligned} \mathfrak{m}_s &= \{v \in \mathfrak{m}^{\mathbb{C}} : \text{Ad}(e^{ts})v \text{ is bounded as } t \rightarrow \infty\} \\ \mathfrak{m}_s^0 &= \{v \in \mathfrak{m}^{\mathbb{C}} : \text{Ad}(e^{ts})v = v \text{ for every } t\}. \end{aligned}$$

One has that \mathfrak{m}_s is invariant under the action of P_s and \mathfrak{m}_s^0 is invariant under the action of L_s . If G is complex, $\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}$ and ι is the adjoint representation, then $\mathfrak{m}_s = \mathfrak{p}_s$ and $\mathfrak{m}_s^0 = \mathfrak{l}_s$.

We need to consider the subalgebra \mathfrak{h}_ι defined as follows. Consider the decomposition $\mathfrak{h} = \mathfrak{z} + [\mathfrak{h}, \mathfrak{h}]$, and the isotropy representation $d\iota : \mathfrak{h} \rightarrow \text{End}(\mathfrak{m})$. Let $\mathfrak{z}' = \ker(d\iota|_{\mathfrak{z}})$ and take \mathfrak{z}'' such that $\mathfrak{z} = \mathfrak{z}' + \mathfrak{z}''$. Define the subalgebra $\mathfrak{h}_\iota := \mathfrak{z}'' + [\mathfrak{h}, \mathfrak{h}]$. The subindex ι denotes that we have taken away the part of the centre \mathfrak{z} acting trivially via the isotropy representation $d\iota$.

Let (E, ϕ) be parabolic G -Higgs bundle over (X, D) . Let \mathfrak{z} be the Lie algebra of $Z(H)$, and let $c \in \sqrt{-1}\mathfrak{z}$. We say that (E, ϕ) is c -**polystable** if for every $s \in \sqrt{-1}\mathfrak{h}$ and any holomorphic reduction of the structure group of E to P_s , σ , such that $\phi|_{X \setminus D} \in H^0(X \setminus D, E_\sigma(\mathfrak{m}_s) \otimes K)$, where E_σ is the principal P_s -bundle obtained from the reduction σ , we have

$$\text{pardeg } E(\sigma, \chi_s) - \langle c, s \rangle \geq 0;$$

and moreover equality occurs only if there is a holomorphic reduction σ_L of the structure group to L_s and a reduction of the parabolic structure to the reduced L_s -bundle E_{σ_L} , so that $\phi|_{X \setminus D} \in H^0(X \setminus D, E_{\sigma_L}(\mathfrak{m}_s^0) \otimes K)$.

Remark 3.2. If the weights are in $\sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$ we can define the sheaf $PE_\sigma(\mathfrak{m}_s)$ of parabolic sections of $E_\sigma(\mathfrak{m}_s)$ as the sheaf of holomorphic sections ψ of $E_\sigma(\mathfrak{m}_s)$ such that $\psi(x_i) \in \mathfrak{m}_i \cap E_\sigma(\mathfrak{m}_s)_{x_i}$, and require that $\phi \in H^0(X, PE_\sigma(\mathfrak{m}_s) \otimes K_{\log D})$.

4. HITCHIN-KOBAYASHI CORRESPONDENCE

Let X be a compact Riemann surface and let D a divisor of X as above. Choose a smooth 2-form ω on X , or exploding at the parabolic points less rapidly than the Poincaré metric (given by 4.16), and normalized such that $\int \omega = 2\pi$. Let (E, ϕ) be a parabolic G -Higgs bundle on (X, D) . Let $c \in \sqrt{-1}\mathfrak{z}$, as in §3.2. We are looking

for a metric h on E outside the divisor D , i.e. $h \in \Gamma(X \setminus D, E(H \setminus H^{\mathbb{C}}))$, satisfying the c -Hermite-Einstein equation:

$$(4.1) \quad R(h) - [\phi, \tau_h \phi] + \sqrt{-1}c\omega = 0,$$

where $R(h)$ is the curvature of the unique connection $A(h)$ compatible with the holomorphic structure of E and the metric h , and τ_h is the conjugation on $\Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}}))$ defined by combining the metric h and the standard conjugation on X from $(1, 0)$ -forms to $(0, 1)$ -forms. We shall denote

$$F(h) = R(h) - [\phi, \tau_h(\phi)] + \sqrt{-1}c\omega.$$

4.1. Initial metric. We begin by constructing a singular metric h_0 which gives an approximate solution to the equations. This metric is α -adapted, in the sense of §2.5.

We first construct a model metric near each singular point x_i . We decompose $\text{Gr Res}_{x_i} \phi$ into its semisimple part and its nilpotent part,

$$(4.2) \quad \text{Gr Res}_{x_i} \phi = s_i + Y_i.$$

Let $e_i \in E_{x_i}$ be an element belonging to the P_{α_i} orbit specified by the parabolic structure. Choose a local holomorphic coordinate z , and extend the trivialization e_i into a holomorphic trivialization of E around x_i , so we can identify locally the metric with a map into $H \setminus H^{\mathbb{C}}$. If $Y_i = 0$, then the model metric is

$$(4.3) \quad h_0 = |z|^{-2\alpha_i} (= e^{-2\alpha_i \ln |z|}).$$

If we change the trivialization by a gauge transformation $g \in \Gamma(X, PE(H^{\mathbb{C}}))$, the resulting metric will remain at bounded distance of h_0 in $H \setminus H^{\mathbb{C}}$, so h_0 defines a *quasi-isometry* class of metrics on E near x_i .

If $Y_i \neq 0$, then consider the reductive subalgebra

$$(4.4) \quad \mathfrak{r} = \ker(\text{Ad}(e^{2\pi\sqrt{-1}\alpha_i}) - 1) \cap \ker(\text{ad } s_i)$$

of $\mathfrak{g}^{\mathbb{C}}$. Recall that $\text{Gr Res}_{x_i} \phi$ is the projection of the residue of ϕ to the centralizer of $e^{2\pi\sqrt{-1}\alpha_i}$, so Y_i belongs to \mathfrak{r} . We can complete Y_i into a Kostant-Rallis \mathfrak{sl}_2 -triple (H_i, X_i, Y_i) (see Appendix C) such that

$$(4.5) \quad H_i \in \mathfrak{h}^{\mathbb{C}} \cap \mathfrak{r}, \quad X_i, Y_i \in \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{r}.$$

Moreover, maybe after conjugation (which means that we change the trivialization), we can assume that

$$(4.6) \quad H_i \in \sqrt{-1}\mathfrak{h}, \quad X_i = -\tau(Y_i),$$

where τ is the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to a compact form so that θ and τ commute. Now the Higgs field has the form

$$(4.7) \quad \phi = (s_i + \text{Ad}(z^{\alpha_i})(Y_i + O(z))) \frac{dz}{z}$$

(this is well defined because $Y_i \in \ker(\text{Ad}(e^{2\pi\sqrt{-1}\alpha_i}) - 1)$ so that Y_i decomposes on eigenspaces of $\text{ad } \alpha_i$ corresponding to the integral eigenvalues). The model for the metric is

$$(4.8) \quad h_0 = |z|^{-2\alpha_i} (-\ln |z|)^{H_i}$$

Again, a different choice of trivialization leads to a metric which remains quasi-isometric to h_0 .

Now extend the local model to some global metric h_0 on E . The quasi-isometry class of h_0 is well-defined. We shall now prove that the metric h_0 gives an approximate solution of the Hermite-Einstein equation near the marked point x_i . Instead of working in the holomorphic gauge e of E , we choose the unitary gauge

$$(4.9) \quad \mathbf{e} = eh_0^{-1/2} = e|z|^{\alpha_i}(-\ln|z|^2)^{-H_i/2},$$

so that the $\bar{\partial}$ -operator of E can be written locally as

$$(4.10) \quad \bar{\partial}^E = \bar{\partial} + h_0^{1/2} \bar{\partial} h_0^{-1/2} = \bar{\partial} + \left(\alpha_i - \frac{H_i}{\ln|z|^2}\right) \frac{d\bar{z}}{2\bar{z}}$$

and the associated H -connection is

$$(4.11) \quad A_0 = d - \sqrt{-1} \left(\alpha_i - \frac{H_i}{\ln|z|^2}\right) d\theta.$$

On the other hand, still in the unitary gauge \mathbf{e} , we obtain the expression for the Higgs field by the action of $\text{Ad}(h_0^{1/2})$, which gives

$$(4.12) \quad \phi = \left(s_i - \frac{\text{Ad}(e^{\sqrt{-1}\theta\alpha_i})Y_i}{\ln|z|^2}\right) \frac{dz}{z} + O(|z|^\epsilon) \frac{dz}{z}.$$

As in formula (4.7) this is well defined. Therefore we obtain, using (4.6),

$$(4.13) \quad \begin{aligned} R(h_0) &= H_i \frac{dz \wedge d\bar{z}}{|z|^2(\ln|z|^2)^2}, \\ [\phi, \tau_{h_0}(\phi)] &= H_i \frac{dz \wedge d\bar{z}}{|z|^2(\ln|z|^2)^2} + O(|z|^\epsilon) \frac{dz \wedge d\bar{z}}{|z|^2}, \end{aligned}$$

and we clearly have an approximate solution of the Hermite-Einstein equation (4.1).

We can regard the unitary gauge \mathbf{e} as defining a different extension of E near the punctures, which is a unitary extension. The resulting H -principal bundle on X will be denoted \mathbf{E} .

4.2. The correspondence. We are now in a position to state the main theorem in this section. First remark that if χ is a character of G , then $\chi([\phi, \theta(\phi)]) = 0$ and therefore the Hermite-Einstein equation (4.1) implies $\text{pardeg}_\chi E = \chi(c)$. In particular, if $c = 0$, $\text{pardeg}_\chi E = 0$. It is important to note that this is no longer true in general for a character of H alone, so we cannot conclude that the total parabolic degree of E must vanish. This justifies the topological condition in the following theorem.

Theorem 4.1. *Let (E, ϕ) be a parabolic G -Higgs bundle, equipped with an adapted initial metric h_0 . Suppose that $\text{pardeg}_\chi E = \chi(c)$ for all characters of \mathfrak{g} . Then (E, ϕ) admits a Hermite-Einstein metric h , quasi-isometric to h_0 , if and only if (E, ϕ) is c -polystable.*

It seems difficult to reduce Theorem 4.1 to the theorem of Simpson [42] for the case $G = GL_n\mathbb{C}$ by taking a faithful representation, since in particular it is not clear how the stability conditions would relate. Instead, we prefer to give a direct proof by checking that the proof in [7] still applies here.

The necessity of the stability condition is a direct consequence of Lemma 2.12, since the second term in the degree is nonnegative for a holomorphic reduction.

We now prove that it is sufficient. For clarity, we restrict to the case $c = 0$ (it is well-known how to modify the proof to handle nonzero c , see for example [14]). Recall that there is a Donaldson functional $M(h, h')$, defined for two metrics h and

h' on E (which we shall take quasi-isometric to h_0), such that

$$(4.14) \quad M(h, h'') = M(h, h') + M(h', h'')$$

$$(4.15) \quad \frac{d}{dt} M(h, h e^{ts}) \Big|_{t=0} = \int_X K(\sqrt{-1}F(h), s).$$

(No boundary term shall occur at the punctures.) In particular the critical points of M are the Hermite-Einstein metrics.

The method consists in minimizing the Donaldson functional $M(h_0, h)$ in a space of metrics h with a priori good control near the punctures (and in particular h remaining quasi-isometric to h_0). The polystability enables to prove C^0 -convergence of a minimizing sequence, and then convergence in stronger functional spaces follows. Finally the limit is proved to be a solution of the Hermite-Einstein equation.

We shall not give full details since the proof follows the one in [7], and we refer to this reference for details. We give only the general setup.

The equation to solve does not depend on the metric on the Riemann surface. Because of the calculation (4.13), it is natural to work with a cusp metric near the punctures, that is equal to

$$(4.16) \quad ds^2 = \frac{|dz|^2}{|z|^2 \ln^2 |z|^2}$$

in some fixed local coordinate z near each puncture. Writing $z = |z|e^{\sqrt{-1}\theta}$ and $t = \ln(-\ln |z|^2)$, this can be written

$$ds^2 = dt^2 + e^{-2t} d\theta^2.$$

Extend t by a smooth function in the interior of X . We define weighted C^0 and L^p spaces by

$$C_\delta^0 = e^{-\delta t} C^0, \quad L_\delta^p = e^{-(\delta + \frac{1}{p})t} L^p.$$

The curious choice for L^p is motivated by the compatibility with C^0 : indeed, with this choice, we have $C_\delta^0 \subset L_{\delta'}^p$ as soon as $\delta > \delta'$, since $\text{vol} = e^{-t} dt d\theta$ near the punctures. The exponent p is thought as being very large—a replacement of ∞ because C^k spaces are not suitable for elliptic analysis.

Consider the H -connection ∇^+ induced by h_0 on E , and define

$$\nabla = \nabla^+ \oplus \text{ad}(\phi) : \mathbf{E}(\mathfrak{g}^{\mathbb{C}}) \rightarrow \Omega^1 \otimes (\mathbf{E}(\mathfrak{g}^{\mathbb{C}}) \oplus \mathbf{E}(\mathfrak{g}^{\mathbb{C}})).$$

Define now the weighted spaces C_δ^k (resp. the weighted Sobolev spaces $L_\delta^{k,p}$) of sections f of $\mathbf{E}(\mathfrak{g}^{\mathbb{C}})$ such that $\nabla^j f \in C_\delta^0$ (resp. L_δ^p) for $j \leq k$. We will also use the refinement $\hat{C}_\delta^k(E(\mathfrak{g}^{\mathbb{C}}))$ (resp. $\hat{L}_\delta^{k,p}(\mathbf{E}(\mathfrak{g}^{\mathbb{C}}))$) of sections f of $\mathbf{E}(\mathfrak{h}^{\mathbb{C}})$ such that $\nabla f \in C_\delta^{k-1}$ (resp. $L_\delta^{k-1,p}$), but we ask nothing on f itself.

Recall from (4.4) the subalgebra

$$(4.17) \quad \mathfrak{r}_i = \ker(\text{Ad}(e^{2\pi\sqrt{-1}\alpha_i}) - 1) \cap \ker(\text{ad } s_i),$$

and define furthermore \mathfrak{r}'_i the commutator of the \mathfrak{sl}_2 -triple (H_i, X_i, Y_i) ,

$$(4.18) \quad \mathfrak{r}'_i = \ker(\text{ad } H_i) \cap \ker(\text{ad } X_i) \cap \ker(\text{ad } Y_i).$$

For $\delta > 0$, it is easy to see that, near a puncture, an element f of \hat{C}_δ^k (resp. $\hat{L}_\delta^{k,p}(\mathfrak{g}^{\mathbb{C}})$) can be decomposed as

$$f = \text{Ad}(e^{\sqrt{-1}\theta\alpha_i})f(0) + f_1, \quad f(0) \in \mathfrak{r}_i \cap \mathfrak{r}'_i, \quad f_1 \in C_\delta^k \text{ (resp. } L_\delta^{k,p}).$$

As before, this is well defined since $f(0) \in \ker(\text{Ad}(e^{2\pi\sqrt{-1}\alpha_i}) - 1)$. Therefore the elements of \hat{C}_δ^2 or $\hat{L}_\delta^{2,p}$ do not go to zero at the punctures. Furthermore one checks easily that $\hat{L}_\delta^{2,p}(\mathbf{E}(\mathfrak{g}^{\mathbb{C}}))$ is a Lie algebra.

Finally the space of metrics in which we look for a solution of the Hermite-Einstein equation is, for a small $\delta > 0$ and a large p ,

$$(4.19) \quad \mathcal{H} = \{h = h_0 e^s, s \in \hat{L}_\delta^{2,p}(\mathbf{E}(\sqrt{-1}\mathfrak{h}))\}.$$

From equation (4.13) it is clear that $F(h_0) \in L_\delta^{2,p}$ for any p and any $\delta > 0$. We choose any fixed $\delta \in (0, 1)$.

4.3. Donaldson's functional. For a pair of metrics $h_0, h_0 e^s \in \mathcal{H}$ let

$$\begin{aligned} M(h_0, h_0 e^s) &= \sqrt{-1} \int_X \text{Tr}(s \Lambda F(h_0)) + \\ &\quad + \int_0^1 (1-t) \|e^{\sqrt{-1}ts/2} D''(s) e^{-\sqrt{-1}ts/2}\|_{L^2(X)}^2 dt, \end{aligned}$$

where $D''(s) = \bar{\partial}^E(s) + [\phi, s]$. Note that $\bar{\partial}^E(s)$ belongs to $\Omega^{0,1}(E(\mathfrak{h}^\mathbb{C}))$ and $[\phi, s]$ belongs to $\Omega^{1,0}(E(\mathfrak{m}^\mathbb{C}))$. Hence the two summands are orthogonal and we can write

$$\begin{aligned} \|e^{\sqrt{-1}ts/2} D''(s) e^{-\sqrt{-1}ts/2}\|_{L^2(X)}^2 &= \|e^{\sqrt{-1}ts/2} \bar{\partial}^E(s) e^{-\sqrt{-1}ts/2}\|_{L^2(X)}^2 + \\ &\quad + \|e^{\sqrt{-1}ts/2} [\phi, s] e^{-\sqrt{-1}ts/2}\|_{L^2(X)}^2. \end{aligned}$$

This allows to view M as a particular case of the functionals defined in [14, 33] using a symplectic point of view (they are instances of integrals of a moment map). In particular, the arguments given in [op.cit.] imply that M satisfies the cocycle condition (4.14) and property (4.15).

The functional M is also an analogue of Donaldson's functional considered by Simpson in [41]. Indeed, we have the following formula

$$\int_0^1 (1-t) \|e^{\sqrt{-1}ts/2} D''(s) e^{-\sqrt{-1}ts/2}\|_{L^2(X)}^2 dt = \int_X \langle \psi(s)(D''s), D''s \rangle_{h_0},$$

where we apply here the notation introduced in §2.4 to the adjoint and the isotropy representations, and extend it to sections of twists of $E(\mathfrak{h}^\mathbb{C})$ and $E(\mathfrak{m}^\mathbb{C})$, for the function $\psi(t) = (e^t - t - 1)/t^2$. Hence Donaldson's functional can be rewritten in the following form

$$M(h_0, h_0 e^s) = \sqrt{-1} \int_X \text{Tr}(s \Lambda F(h_0)) + \int_X \langle \psi(s)(D''s), D''s \rangle_{h_0},$$

which makes evident the relation to Simpson's definition.

4.4. The proof of the correspondence. The method consists in minimizing the Donaldson functional $M(h_0, h)$ for $h \in \mathcal{H}$ under the constraint $\|F(h)\|_{L_\delta^{2,p}} \leq B$ for some large B . The technical results in [7] reduce this problem to obtaining a C^0 -estimate on a minimizing sequence h_i , which in turn is a consequence of polystability, as we shall now see.

At the end, the solution h satisfies an elliptic equation, so has additional regularity:

$$(4.20) \quad h \in \mathcal{H}^\infty = \{h_0 e^s, s \in \hat{C}_\delta^\infty(\mathbf{E}(\sqrt{-1}\mathfrak{h}))\}.$$

Actually it can be shown that there is a stronger decay of the components of s which are orthogonal to \mathfrak{r}_i .

Take some big B and define

$$\mathcal{S}^\infty(B) = \{s \in \hat{C}_\delta^\infty(\mathbf{E}(\sqrt{-1}\mathfrak{h})), \|F(h_0 e^s)\|_{L_\delta^{2,p}} \leq B\}.$$

Assume that there do not exist any constants C, C' such that

$$\sup |s| \leq C + C' M(h_0, h_0 e^s) \quad \text{for any } s \in \mathcal{S}^\infty(B).$$

The bound on the curvature implies that

$$\sup |s| \leq \text{const.}(1 + \|s\|_{L^1})$$

for some constant depending on B (see Lemma 8.4 in [7]). Hence our assumption implies that there neither exist constants C, C' such that

$$\|s\|_{L^1} \leq C + C'M(h_0, h_0 e^s) \quad \text{for any } s \in \mathcal{S}^\infty(B).$$

It follows that we can take a sequence $\{s_i\} \subset \mathcal{S}^\infty(B)$ and positive real numbers $C_i \rightarrow \infty$ such that

$$|s_i|_{L^1} \rightarrow \infty, \quad C_i M(h_0, h_0 e^{s_i}) \leq |s_i|_{L^1} =: l_i.$$

Define $u_i = l_i^{-1} s_i$, so that $\|u_i\|_{L^1} = 1$. Now the arguments in §5 of [41] imply that, up to passing to a subsequence, the sections u_i converge (weakly and locally in $L^{1,2}$) to a nonzero and locally $L^{1,2}$ section u_∞ of $\mathbf{E}(\sqrt{-1}\mathfrak{h})|_{X \setminus D}$. This section satisfies the following properties:

- the L^2 norm of $D''u_\infty$ is finite;
- there exist real numbers $l_1 < \dots < l_k$ and locally $L^{1,2}$ sections π_1, \dots, π_k of $\text{End}(E(\mathfrak{h}^\mathbb{C}))|_{X \setminus D}$ such that $\text{ad}(u_\infty) = \sum l_j \pi_j$;
- for each j let $\Pi^j = \pi_1 + \dots + \pi_j$. Then $(1 - \Pi^j)D''\Pi^j = 0$.

The regularity result of Uhlenbeck and Yau for locally $L^{1,2}$ subbundles implies that the image of Π^j is a holomorphic subbundle $E^j \subset E(\mathfrak{h}^\mathbb{C})|_{X \setminus D}$ (see Proposition 5.8 in [41]). This implies that u_∞ is a smooth section of $\mathbf{E}(\sqrt{-1}\mathfrak{h})|_{X \setminus D} \subset E(\mathfrak{h}^\mathbb{C})|_{X \setminus D}$, since h_0 is smooth on $X \setminus D$.

Lemma 4.2. *Take any $x, y \in X \setminus D$, choose trivializations of \mathbf{E}_x and \mathbf{E}_y , and use them to identify $\mathbf{E}(\mathfrak{h}^\mathbb{C})_x$ and $\mathbf{E}(\mathfrak{h}^\mathbb{C})_y$ with $\mathfrak{h}^\mathbb{C}$. Then the elements $u_\infty(x), u_\infty(y) \in \mathfrak{h}^\mathbb{C}$ are conjugate.*

Proof. Denote by $\mathfrak{h}^\mathbb{C} // H^\mathbb{C}$ the affine quotient of the adjoint action. Since semisimple elements have closed adjoint orbits, two semisimple elements $x, y \in \mathfrak{h}^\mathbb{C}$ are conjugate if and only if their images by the projection map $\mathfrak{h}^\mathbb{C} \rightarrow \mathfrak{h}^\mathbb{C} // H^\mathbb{C}$ coincide. The later is equivalent to showing that for any invariant polynomial $p \in \text{Hom}_\mathbb{C}(S^* \mathfrak{h}^\mathbb{C}, \mathbb{C})^{H^\mathbb{C}}$ we have $h(x) = h(y)$. Let p be any such invariant polynomial. The properties of u_∞ stated above imply

$$(4.21) \quad \lim_{t \rightarrow \infty} \|e^{\sqrt{-1}tu_\infty/2} (\bar{\partial}^E u_\infty) e^{\sqrt{-1}tu_\infty/2}\|_{L^2} = 0.$$

This implies that $\bar{\partial}(p(u_\infty)) = 0$, hence $p(u_\infty)$ is a holomorphic function. On the other hand using the isomorphisms

$$\begin{aligned} \text{Hom}_\mathbb{C}(S^* \mathfrak{h}^\mathbb{C}, \mathbb{C})^{H^\mathbb{C}} &\simeq \text{Hom}_\mathbb{C}(S^* \mathfrak{h}^\mathbb{C}, \mathbb{C})^H \quad (H \subset H^\mathbb{C} \text{ is Zariski dense}) \\ &\simeq \text{Hom}_\mathbb{R}(S^*(\sqrt{-1}\mathfrak{h}), \mathbb{R})^H \otimes_\mathbb{R} \mathbb{C} \end{aligned}$$

we may write $p = a + \sqrt{-1}b$, where $a(\sqrt{-1}\mathfrak{h}) \subset \mathbb{R}$ and $b(\sqrt{-1}\mathfrak{h}) \subset \mathbb{R}$ and both a and b are $H^\mathbb{C}$ invariant. Then $a(u_\infty)$ and $b(u_\infty)$ are holomorphic functions and since u_∞ is a section of $\mathbf{E}(\sqrt{-1}\mathfrak{h})$ the function $a(u_\infty)$ (resp. $b(u_\infty)$) is real (resp. imaginary) valued. Hence both $a(u_\infty)$ and $b(u_\infty)$ vanish. We have thus proved that $p(u_\infty) = 0$ for any invariant polynomial p . Since u_∞ takes semisimple values, the lemma is proved. \square

Let $x \in X \setminus D$ be any point, take a trivialization of \mathbf{E}_x , use it to identify $\mathbf{E}(\sqrt{-1}\mathfrak{h})_x$ with $\sqrt{-1}\mathfrak{h}$, and let $s \in \sqrt{-1}\mathfrak{h}$ be the element corresponding to $u_\infty(x)$. Then the section u_∞ defines a reduction σ of the structure group of $E|_{X \setminus D}$ to the parabolic subgroup $P = P_s$. Furthermore, this section is holomorphic, because the bundles $E^j \subset E(\mathfrak{h}^\mathbb{C})$ are holomorphic.

Lemma 4.3. *The reduction σ extends into a holomorphic reduction of E to P on the whole X , such that $\phi \in H^0(X \setminus D, E_\sigma(\mathfrak{m}_s) \otimes K)$.*

The lemma will be proved in §4.5.

The element $s \in \sqrt{-1}\mathfrak{h}$ which was used to define $P = P_s$ also provides a strictly antidominant character $\chi : \mathfrak{p} \rightarrow \mathbb{C}$, see appendix D.1. The same arguments as in Lemma 5.4 in [41] allow to prove that the section u_∞ satisfies

$$\begin{aligned} & \int_{X \setminus D} \langle \Lambda F(h_0), u_\infty \rangle + \int_{X \setminus D} \langle \psi(u_\infty)(\bar{\partial} u_\infty), \bar{\partial} u_\infty \rangle \leq \\ & \leq \int_{X \setminus D} \langle \Lambda F(h_0), u_\infty \rangle + \int_{X \setminus D} \langle \psi(u_\infty)(D'' u_\infty), D'' u_\infty \rangle \leq 0. \end{aligned}$$

This contradicts polystability. since by Lemma 2.12 the first expression above is $2\pi \text{pardeg}(E)(\sigma, \chi)$.

4.5. Extending σ to the whole X . Here we prove Lemma 4.3. The proof is a bit long, and instead one would like to reduce to the case of a line bundle as in Simpson [41, Lemma 10.2], but it is not obvious how to make this work here: the problem is that, a priori, the character of \mathfrak{p} appearing in the proof does not lift to P , so one cannot appeal to Borel-Weil-Bott theory (this is the Plücker embedding used by Simpson). So instead we prove the Lemma directly.

Since the question is purely local we concentrate our attention to any point $x \in X$ belonging to the support of the divisor D with parabolic weight α . First we observe that we can reduce the situation to the case in which $\alpha \in \sqrt{-1}\mathcal{A}'$. To see this, suppose that $\alpha \in \sqrt{-1}(\bar{\mathcal{A}} \setminus \mathcal{A}')$. We can apply Proposition 2.5 to find a ramified cover p over x such that by applying Hecke and Weyl transformations the resulting parabolic weight of p^*E on $y = p^{-1}(x)$ be in $\sqrt{-1}\mathcal{A}'$. Now, a reduction of E to a parabolic subgroup P (in the neighbourhood of x) gives reduction of p^*E to P , but by Lemma 2.7 this defines a reduction to P of a Hecke and Weyl transform of p^*E so that the parabolic weight of p^*E at y is in $\sqrt{-1}\mathcal{A}$. We can then do our analysis on the local cover under the desired condition that the parabolic weights are in \mathcal{A}' .

Let $\Delta \subset X$ be a disc centered at x and choose a holomorphic identification $\Delta^* := \Delta \setminus \{x\} \simeq S := (0, \infty) \times S^1$, so that the coordinates (t, θ) give polar coordinates $(r = e^{-t}, \theta)$ in Δ . Trivializing $E|_{\Delta^*}$ by means of the unitary gauge \mathbf{e} we can look at the reduction $\sigma|_{\Delta^*}$ as a map $\psi : S \rightarrow F := H^\mathbb{C}/P$. Then ψ is holomorphic: $\bar{\partial}^E \psi = 0$, and it has finite energy: $\|d_{A_0} \psi\|_{L^2}^2 < \infty$ (here we assume that F is endowed with an arbitrary H -invariant Kähler metric).

Assume that $x = x_i$ and let $\alpha := \alpha_i$ and $\xi := H_i$ be the two elements in $\sqrt{-1}\mathfrak{h}$ used to construct the model metric h_0 near x_i (see §4.1). Let V_α, V_ξ be the vector fields on F defined by the infinitesimal action of α and ξ respectively. Using (4.10) the equation $\bar{\partial}^E \psi = 0$ may be written as:

$$(4.22) \quad \partial_t \psi = I(\psi) \partial_\theta \psi + V_\alpha(\psi) - \frac{1}{2t} V_\xi(\psi),$$

where I denotes the complex structure on F and $\partial_t, \partial_\theta$ denote partial derivatives. Define the energy (half)density function

$$E(t, \theta) = |\partial_t \psi(t, \theta)| = |I(\psi(t, \theta)) \partial_\theta \psi(t, \theta) + V_\alpha(\psi(t, \theta)) - \frac{1}{2t} V_\xi(\psi(t, \theta))|.$$

That ψ has finite energy translates into

$$(4.23) \quad \int E(t, \theta)^2 dt d\theta < \infty.$$

Lemma 4.4. *For any θ we have $\lim_{t \rightarrow \infty} E(t, \theta) = 0$ uniformly on θ .*

Proof. We use standard arguments in the theory of (pseudo)holomorphic maps. We first prove that E is bounded. Assume that this is not the case, and let $s_j = (t_j, \theta_j)$ be a sequence such that $e_j := E(s_j) \rightarrow \infty$. Restricting ψ to the ball $B(s_j, e_j^{-1/2})$ and rescaling by a factor of e_j we get a map $g_j : B(0, e_j^{1/2}) \rightarrow F$, where $B(0, e_j^{1/2}) \subset \mathbb{C}$ is the disk centered at 0 of radius $e_j^{1/2}$. We can assume that $\sup |dg_j|$ is attained at 0 and hence is equal to 1 (otherwise we replace s_j by a suitable point in $B(s_j, e_j^{-1/2})$). Then, taking a subsequence if necessary, the maps g_j converge in the weak C^∞ -topology to a holomorphic map $g : \mathbb{C} \rightarrow F$, since the terms $V_\alpha - (2t)^{-1}V_\xi$ in (4.22) disappear in the limit due to the rescaling factor. Furthermore, the energy of g is a positive number $\eta > 0$, since we have $|dg(0)| = 1$. By construction we also have

$$\liminf_{j \rightarrow \infty} \int_{B(s_j, e_j^{-1/2})} |\partial_t \psi|^2 dt d\theta = \eta > 0,$$

which contradicts the finite energy condition (4.23).

We now prove the statement of the lemma, again by contradiction. Assume that $s_j = (t_j, \theta_j)$ is a sequence of points in S and that there exists some $\delta > 0$ such that $E(s_j) \geq \delta$ for each j . Consider the map

$$h_j : (-1, 1) \times S^1 \rightarrow F$$

given by $h_j(\tau, \theta) = \psi(t_j + \tau, \theta)$. By our previous arguments, the first derivative of h_j is uniformly bounded, hence extracting a subsequence if necessary we may assume that the maps h_j converge in the weak C^∞ topology to a holomorphic map $h : (-1, 1) \times S^1 \rightarrow F$. We may also assume that $\theta_j \rightarrow \theta$. Then, by the choice of s_j we must have $|dh(0, \theta)| \geq \delta > 0$. But the finite energy condition (4.23) forces $\int_{(-1, 1) \times S^1} |dh|^2 d\tau d\theta = 0$, which is a contradiction. \square

Let $F_\alpha \subset F$ be the vanishing locus of V_α . This is a complex submanifold. Indeed, if $T_\alpha \subset H$ denotes the closure of $\{\exp(\sqrt{-1}t\alpha) \mid t \in \mathbb{R}\}$, then T_α is a compact Lie group and F_α coincides with the fixed point set of T_α , which by standard theory of group actions is a smooth complex submanifold. Define similarly the submanifold $F_\xi \subset F$ and the group $T_\xi \subset H$.

Lemma 4.5. *The set F_α coincides with the set of points fixed by $e_\alpha := \exp(2\pi\sqrt{-1}\alpha)$.*

Proof. This is a consequence of the fact that the eigenvalues of $\text{ad}(\alpha)$ have absolute value < 1 . \square

Lemma 4.6. *For any $\epsilon > 0$ there exists some t_0 such that if $t \geq t_0$ then for any θ the distance from $\psi(t, \theta)$ to F_α is $< \epsilon$.*

Proof. Let $\epsilon > 0$ be any number and let F^ϵ be the points of F at distance $\geq \epsilon$ from F_α . Since F_α is the fixed point set of e_α and F^ϵ is compact, there exists some $\delta_1 > 0$ such that for any $p \in F^\epsilon$ we have $d(p, e_\alpha p) \geq \delta_1$, where $d(\cdot, \cdot)$ denotes the distance function of points in F . Using the compactness of F we deduce the following: there exists some $\delta_2 > 0$ such that if $\gamma : [0, 2\pi] \rightarrow F$ is a map satisfying everywhere $|\gamma' + V_\alpha(\gamma)| \leq \delta_2$ then

$$d(\gamma(0), \gamma(2\pi)) \geq d(\gamma(0), e_\alpha \gamma(0)) - \delta_1/2.$$

So, if we additionally have $\gamma(0) \in F^\epsilon$, then

$$d(\gamma(0), \gamma(2\pi)) \geq \delta_1/2.$$

By Lemma 4.4 there exists some t_0 such that for any $t \geq t_0$ and any θ we have $|E(t, \theta)| + (2t)^{-1}|V_\xi(\psi(t, \theta))| \leq \delta_2$ (note that since F is compact there is a uniform upper bound for $|V_\xi(\psi(t, \theta))|$). Applying the previous considerations to $\gamma(a) =$

$\psi(t, e^{2\pi a}\theta)$ and using the fact that $\gamma(0) = \gamma(2\pi)$ we conclude that $\psi(t, \theta) \notin F^\epsilon$, so $\psi(t, \theta)$ is at distance $< \epsilon$ from F_α . \square

Recall that the elements α_i, H_i were chosen in Subsection 4.1 so that the adjoint action of $e^{2\pi\sqrt{-1}\alpha_i}$ fixes H_i . The same arguments as in Lemma 4.5 allow to deduce from it that $[\alpha, \xi] = 0$. In particular the elements in T_α and T_ξ commute. This implies that the vector field V_ξ is tangent to F_α and that the action of T_ξ on F fixes F_α . This is relevant in the proof of the next lemma.

Lemma 4.7. *There exists some $\epsilon, \delta > 0$ such that any ball $B(q, \delta) \subset F^\epsilon$ is contained in a holomorphic coordinate chart $z : (z_1, \dots, z_k) : U \rightarrow \mathbb{C}$, whose image is a ball centered at 0, such that:*

- (1) *either there exist real numbers $a_1, \dots, a_k \in (-1, 1)$ and $b_1, \dots, b_k \in \mathbb{R}$ such that*

$$dz_*(V_\alpha) = \sum_{j=1}^k a_j z_j \partial_{z_j}, \quad dz_*(V_\xi) = \sum_{j=1}^k b_j z_j \partial_{z_j},$$

- (2) *or there exist real numbers $a_1, \dots, a_{k-1} \in (-1, 1)$ and $b \in \mathbb{R}$ such that*

$$dz_*(V_\alpha) = \sum_{j=1}^{k-1} a_j z_j \partial_{z_j}, \quad dz_*(V_\xi) = b \partial_{z_k}.$$

Furthermore, there are constants $A \in (0, 1)$ and $B, C \in \mathbb{R}_{>0}$ depending only on F and its Riemannian metric such that:

- (P1) $|a_j| \leq A$ for each j ,
(P2) in case (1) above we have: for each j , $|b_j| \leq B$, and if $b_j \neq 0$ then $|b_j| \geq B^{-1}$; in case (2) we have $|b| < B$,
(P3) let $|\cdot|_0$ denote the standard norm in \mathbb{C}^n , and let $f : TF|_U \rightarrow \mathbb{R}_{>0}$ be the function defined by the property that for any tangent vector $v \in TF|_U$ we have $|v| = f(v)|dz(v)|_0$; then we have

$$C^{-1} \leq f \leq C, \quad |\nabla f| \leq C, \quad |\nabla^2 f| \leq C.$$

Proof. The proof is left to the reader, we just give a sketch. Construct first a finite number of coordinate charts covering $F_{\alpha, \xi} := F_\alpha \cup F_\xi$ with respect to which the actions of T_α and T_ξ are linear. These charts will be of the first type. They also cover the closure of an open neighborhood \mathcal{U} of $F_{\alpha, \xi}$. Then construct a finite number of equivariant charts of $F_\alpha \setminus \mathcal{U}$ with respect to which T_α is linear and V_ξ is a constant times ∂_{z_k} . Then define ϵ and δ using Lebesgue's lemma for finite coverings of compact spaces. We have $|a_j| < 1$ for each j because α belongs to the Weyl alcove, and the uniform estimates follow from the fact that there are a finite number of charts. \square

Let us define

$$\Xi := I(\psi) \partial_\theta \psi + V_\alpha(\psi)$$

and let

$$\eta(t) := \int |\Xi(t, \theta)|^2 d\theta.$$

Lemma 4.4, together with equation (4.22), implies that

$$(4.24) \quad \int \eta(t) dt < \infty.$$

Lemma 4.8. *There exists some constant $c > 0$ such that, for big enough t ,*

$$(4.25) \quad \eta''(t) \geq c\eta(t).$$

Proof. Take some big t . By Lemma 4.6 we may suppose, provided t is big enough, that there exists a coordinate chart (z_1, \dots, z_k) as the ones given by Lemma 4.7 which contains a neighborhood of $\psi(t, \theta)$ for each $\theta \in S^1$.

Case 1. Assume that we are in the first of the two cases considered in Lemma 4.7. Let $\psi_j = z_j \circ \psi$ be the j -th coordinate of ψ . The holomorphicity equation (4.22) reads

$$\partial_t \psi_j = \sqrt{-1} \partial_\theta \psi_j + (a_j - \frac{1}{2t} b_j) \psi_j.$$

In terms of the Fourier development $\psi_j(t, \theta) = \sum_{l \in \mathbb{Z}} \psi_{jl}(t) \theta^l$ the holomorphicity equation translates into

$$\psi'_{jl} = (-l + a_j - \frac{1}{2t} b_j) \psi_{jl},$$

which can be integrated as:

$$(4.26) \quad \psi_{jl}(t + \tau) = \psi_{jl}(t) e^{(-l + a_j) \tau} \left(\frac{t + \tau}{t} \right)^{-b_j/2}.$$

The j -th coordinate of Ξ is:

$$\Xi_j(t, \theta) = \sqrt{-1} \partial_\theta \psi_j(t, \theta) + a_j \psi(t, \theta).$$

Combining this formula with (4.26) we obtain the following expression for the l -th Fourier coefficient of the Ξ_j :

$$(4.27) \quad \Xi_{jl}(t) = (-l + a_j) \psi_{jl}(t) e^{(-l + a_j) \tau} \left(\frac{t + \tau}{t} \right)^{-b_j/2}.$$

Using the bound $|a_j| \leq A < 1$ (property (P1) in Lemma 4.7) and the fact that l is an integer, it follows that there exists some constant $C_1 > 0$ such that, if t is big enough and $l \neq 0$,

$$(4.28) \quad \frac{\partial^2}{\partial t^2} |\Xi_{jl}(t)|_0^2 \geq C_1 |\Xi_{jl}(t)|_0^2,$$

(recall that $|\cdot|_0$ denotes the standard norm of a vector in \mathbb{C}^k). Furthermore, C_1 is independent of t and of the chosen coordinate chart (by (P1) and (P2) of Lemma 4.7). Define

$$\eta_0(t) = \int |\Xi(t, \theta)|^2 d\theta = \sum_j \int |\Xi_j(t, \theta)|_0^2 d\theta = \sum_{j,l} |\Xi_{jl}(t)|_0^2.$$

It follows from (4.28) that

$$(4.29) \quad \eta_0''(t) \geq C_1 \eta_0(t).$$

To finish the argument we are going to deduce from (4.29) the inequality of the lemma, using (P3) in Lemma 4.7. We have

$$(4.30) \quad \eta(t) = \int f(\Xi)^2 |\Xi|^2 d\theta,$$

which implies that

$$(4.31) \quad C^2 \eta_0(t) \geq \eta(t) \geq C^{-2} \eta_0(t).$$

Deriving twice (4.30) with respect to t we obtain a bound

$$\eta'' \geq C^{-1} (|\nabla^2 \Xi| \eta_0 + |\nabla \Xi| \eta_0' + \eta_0''),$$

where $|\nabla^r \Xi|(t) = \sup_\theta |\nabla^r \Xi(t, \theta)|_0$. Combining Lemma 4.4 with standard elliptic estimates:

$$(4.32) \quad |\nabla^r \Xi(t, \theta)|^2 \leq C(r) \int_{[t-1, t+1] \times S^1} |\Xi|^2 dt d\theta \quad \text{for all } r \geq 1$$

(which can be directly deduced from (4.27), using the fact that $|-l + a_j|$ is either zero or uniformly bounded below by a positive number), it follows that $|\nabla \Xi|$ and $|\nabla^2 \Xi|$ converge to 0 as $t \rightarrow \infty$. On the other hand, (4.27) also implies that

$$|\eta'_0(t)| \leq C_2 \eta''_0(t)$$

for some constant C_2 . It follows that

$$\eta''(t) \geq C^{-1} \eta''_0(t) + o(t)(\eta_0(t) + \eta''_0(t)).$$

Combining this with (4.29) and (4.31) we deduce the statement of the lemma.

Case 2. If our coordinate chart falls in the second case of Lemma 4.7 we can use the same arguments as before involving the Fourier coefficients ψ_{jl} of the coordinates of ψ . Except for the case $(j, l) = (k, 0)$ they satisfy (4.26) with $\beta_l = 0$ for each l and with $\alpha_k = 0$. The only equation which has a different form is $\psi'_{k0} = -(2t)^{-1}$, but this does not affect the computation since ψ_{k0} does not contribute to η_0 . \square

The previous lemma, together with (4.24), implies that η decays exponentially fast: $\eta(t) \leq C e^{-\sigma t}$ for some $\sigma > 0$. Taking into account the elliptic estimates (4.32) we deduce from this that

$$(4.33) \quad |\nabla^r \Xi(t, \theta)| \leq C_1 e^{-\sigma t} \quad \text{for } r = 0, 1, 2.$$

Fix some $\theta_0 \in S^1$ and let $\gamma(t) = \psi(t, \theta_0)$. We have

$$\gamma'(t) = \Xi(\gamma(t)) - \frac{1}{2t} V_\xi(\gamma(t)).$$

Lemma 4.9. *There are constants $C_j, \sigma_j > 0$ so that, for big enough t , we have*

- (1) $|\Xi(\gamma(t))| \leq C_1 e^{-\sigma_1 t}$,
- (2) $|V_\xi(\gamma(t))| \leq C_2 t^{-\sigma_2}$.

Proof. The first inequality is contained in (4.33), so we only have to prove the second inequality. Define $\phi(t) = \gamma(e^t)$. Then we have

$$\phi'(t) = e^t \gamma'(e^t) = -2V_\xi(\phi(t)) + e^t \Xi(\phi(t)).$$

Let us define for convenience $\mathcal{E}(t) := e^t \Xi(\phi(t))$. By (4.33) we can estimate for big enough t

$$|\mathcal{E}(t)| = |e^t \Xi(\phi(t))| = |e^t \Xi(\gamma(e^t))| \leq C_1 e^t e^{-\sigma_1 e^t} \leq e^{-t}.$$

In the remainder of the proof we will always implicitly assume t to be big enough so that the previous estimate holds.

We claim that $|V_\xi(\phi(t))| \rightarrow 0$ as t goes to infinity. To prove this, it will be useful to take into account that V_ξ is the gradient of a function $h : F \rightarrow \mathbb{R}$. Indeed, the action of T_ξ on F is Hamiltonian, so it admits a moment map $\mu_\xi : F \rightarrow \mathfrak{t}_\xi^*$ (with \mathfrak{t}_ξ the Lie algebra of T_ξ), and h can be chosen to be

$h = \langle \mu_\xi, \xi \rangle$. Now define $\nu(t) = h(\phi(t))$ and compute:

$$\begin{aligned} \nu'(t) &= \langle V_\xi, -2V_\xi + \mathcal{E}(t) \rangle = -2|V_\xi|^2 + \langle V_\xi, \mathcal{E}(t) \rangle \\ &\leq -2|V_\xi|^2 + |V_\xi| e^{-t} \leq -2|V_\xi|^2 + C e^{-t}, \end{aligned}$$

where $C = \sup_F |V_\xi|$. This implies that for any δ there exists some t_1 such that $\nu([t_1, \infty)) \subset (-\infty, \nu(t_1) + \delta]$. Since h is bounded (F is compact), the latter property implies the existence of $\lambda = \lim_{t \rightarrow \infty} \nu(t)$, so we deduce that for any δ there exists some t_1 such that

$$(4.34) \quad \nu([t_1, \infty)) \subset [\nu(t_1) - \delta, \nu(t_1) + \delta].$$

Now take any $\epsilon > 0$ and define

$$\eta = \inf \{d(x, y) \mid |V_\xi(x)| = \epsilon/2, |V_\xi(y)| = \epsilon\},$$

where $d(\cdot, \cdot)$ denotes the distance function of points in F . Since F is compact, $\eta > 0$. Using $\phi'(t) = -2V_\xi + \mathcal{E}(t)$, we can estimate $|\phi'(t)| \leq C$ for big enough t , where C is as before. Thinking of this as an upper bound on the speed of $\phi(t)$ it is clear from the definition of η that if $|V_\xi(\phi(t))| \geq \epsilon$ then for any $\tau \in [t, t + \eta C^{-1}]$ we will have $|V_\xi(\phi(\tau))| \geq \epsilon/2$. Integrating the bound on $\nu'(t)$ we obtain

$$\begin{aligned} \nu(t + \eta C^{-1}) &\leq \nu(t) - 2(\epsilon/2)^2 \eta C^{-1} + \int_t^\infty e^{-s} ds \\ &\leq \nu(t) - (\epsilon/2)^2 \eta C^{-1} \end{aligned}$$

for t bigger than some $t_2 > 0$. Taking any $\delta < (\epsilon/2)^2 \eta C^{-1}/2$ and defining t_1 as before, we deduce from (4.34) that t cannot be simultaneously bigger than t_1 and t_2 . Hence setting $t_0 = \sup\{t_1, t_2\}$ we deduce that for any $t \geq t_0$ we have $|V_\xi(\phi(t))| < \epsilon$, which is what we wanted to prove.

Let now $\rho(t) = |V_\xi(\phi(t))|^2$. We claim that there exist constants $C_3 > 0$ and $C_4 > 0$ such that, if t is big enough, then we have

$$(4.35) \quad \rho''(t) \geq C_3 \rho(t) \quad \text{unless} \quad \rho(t) \leq C_4 e^{-t}.$$

Choosing ϵ small enough in the previous claim, we deduce that for any $t \geq t_0$ the point $\phi(t)$ is included in a coordinate chart of the first type as given by Lemma 4.7. Take any such t and pick one of those coordinate charts. Let the map $\phi_0 : (t - \epsilon, t + \epsilon) \rightarrow F$ satisfy

$$\phi_0(t) = \phi(t), \quad \phi_0'(\tau) = -2V_\xi(\phi_0(\tau))$$

and let $\phi_{01}, \dots, \phi_{0k}$ be the coordinates of ϕ_0 . We compute:

$$\rho_0(\tau) := |V_\xi(\phi_0(\tau))|_0^2 = \sum_j b_j^2 |\phi_{0j}(\tau)|^2 e^{-4b_j(\tau-t)}.$$

From this one deduces (using the lower bound on nonzero b_j 's given in (P2) in Lemma 4.7) that the function ρ_0 satisfies

$$\rho_0'' \geq 2C_3 \rho_0$$

for some constant C_3 . To deduce (4.35) from the previous estimate, note that ρ_0 differs from ρ in two things: first, ρ_0 is defined using the norm $|\cdot|_0$ and not the one coming from the Riemannian metric in F ; and, second, in the definition of ϕ_0' we have omitted the term $\mathcal{E}(t)$ which appears in ϕ' . The first problem can be addressed using the same technique as in the proof of Lemma 4.8. The second one is responsible for the conditional assertion in (4.35): roughly speaking, when we pass from ρ_0 to ρ the convexity property survives unless the difference of ρ and ρ_0 (and its first two derivatives) are bigger than $C\rho_0$ for some small constant C . Finally, the difference of ρ and ρ_0 and its derivatives can be estimated using (4.33).

Property (4.35) implies, using an easy argument which is left to the reader (and which is analogous to the arguments in §11.1 in [36]), the following exponential decay: $|V_\xi(\phi(t))| \leq C_2 e^{-\sigma_2 t}$ for some $C_2, \sigma_2 > 0$. Using $\gamma(t) = \phi(\ln t)$ we deduce the required inequality

$$|V_\xi(\gamma(t))| \leq C_2 t^{-\sigma_2}.$$

□

Lemma 4.10. *There exists some $p \in F_\alpha \cap F_\xi$ such that $\psi(t, \theta)$ converges to p as $t \rightarrow \infty$ uniformly on θ .*

Proof. By Lemma 4.9 the function

$$|\gamma'(t)| = |\Xi(\gamma(t)) - \frac{1}{2t} V_\xi(\gamma(t))|$$

is integrable, hence $\gamma(t)$ converges to some point $p \in F$ as $t \rightarrow \infty$. The fact that both $|V_\alpha(\gamma(t))|$ and $|V_\xi(\gamma(t))|$ converge to 0 as $t \rightarrow \infty$ implies that $p \in F_\alpha \cap F_\xi$. The lemma follows from the fact that $|\partial_\theta \psi(t, \theta)|$ converges to 0 as $t \rightarrow \infty$ uniformly on θ , because

$$d(\psi(t, \theta), \gamma(t)) \leq \int_{\theta_0}^{\theta} |\partial_\theta \psi(t, \beta)| d\beta$$

(as before, $d(\cdot, \cdot)$ denotes the distance between points in F). \square

The previous lemma implies that ψ extends continuously at the puncture $0 \in \Delta$. Let us denote by $\Psi : \Delta \rightarrow F$ the resulting map. Our next aim is to prove that the map $\psi^{\mathcal{H}} : S \rightarrow F$, obtained by putting ψ in holomorphic gauge, also extends at the punctures. Furthermore, we want to prove that $p^{\mathcal{H}} := \lim_{t \rightarrow \infty} \psi^{\mathcal{H}}(t, \theta)$ belongs to a gradient curve of V_α converging to p .

It follows from Lemma 4.10 that there is a coordinate chart of the first type in Lemma 4.7 which contains $\psi(t, \theta)$ for $t \geq t_0$. By the proof of Lemma 4.8 the Fourier coefficients of the coordinates of ψ are given by

$$(4.36) \quad \psi_{jl}(t_0 + \tau) = \psi_{jl}(t_0) e^{(-l+a_j)\tau} \left(\frac{t_0 + \tau}{t_0} \right)^{-b_j/2}.$$

Since (ψ_1, \dots, ψ_k) converges to $(0, \dots, 0)$ as $t \rightarrow \infty$ we must have $\psi_{jl} = 0$ whenever $a_j \geq l$. Since $a_j \in (-1, 1)$, it follows in particular that $\psi_{jl} = 0$ if $l < 0$.

We deduce from (4.36) and (4.9) that passing from the unitary to a holomorphic gauge the map ψ gets transformed into a map $\psi^{\mathcal{H}} : S \rightarrow F$ whose coordinate Fourier coefficients are given by

$$\begin{aligned} \psi_{jl}^{\mathcal{H}}(t_0 + \tau) &= e^{-(t_0+\tau)a_j} (2(t_0 + \tau))^{b_j/2} \psi_{jl}(t_0 + \tau) \\ &= \psi_{jl}(t_0) e^{-l\tau} e^{-t_0 a_j} t_0^{b_j/2}. \end{aligned}$$

We have $\sum_{j,l} |\psi_{jl}(t_0)|^2 < \infty$. So, using the identification $S \simeq \Delta^*$, $\psi^{\mathcal{H}}$ extends to a holomorphic map $\Psi^{\mathcal{H}} : \Delta \rightarrow F$. Furthermore, the j -th coordinate of $\Psi^{\mathcal{H}}(0)$ is given in our chart by

$$\Psi_j^{\mathcal{H}}(0) = \psi_{j0}(t_0) e^{-t_0 a_j} t_0^{b_j/2},$$

which vanishes whenever $a_j \leq 0$. This implies that the curve $\gamma : [0, \infty) \rightarrow \mathbb{C}^n$ defined by the properties $\gamma(0) = \Psi^{\mathcal{H}}(0)$ and $\gamma' = V_\alpha(\gamma)$ satisfies $\gamma(t) \rightarrow p$ as $t \rightarrow \infty$. In geometric terms: the integral curve of V_α passing through $\Psi^{\mathcal{H}}(0)$ converges to $\Psi(0)$.

5. PARABOLIC LOCAL SYSTEMS

In this section we take $c = 0$ and we refer to c -(poly)stability of a parabolic G -Higgs bundle simply as (poly)stability.

5.1. From Higgs bundles to parabolic local systems. Let (E, ϕ) be a polystable parabolic G -Higgs bundle with $\text{pardeg}_\chi E = 0$ for all characters of G . By Theorem 4.1, we get an Hermitian-Einstein metric h on E , which is quasi-isometric to some given adapted metric h_0 . Equation 4.1 simply means that

$$D = A(h) + \phi - \sigma_h(\phi)$$

is a flat G -connection on the G -bundle obtained by extending the structure group of the H -bundle given by the Hermite-Einstein metric. We therefore obtain a G -local system on $X' := X \setminus \{x_1, \dots, x_r\}$. Recall that a G -local system on a manifold can be equivalently seen as a representation of the fundamental group of the manifold in G , a G -bundle over the manifold equipped with a flat G -connection, or a G -bundle with locally constant transition functions.

Let us now examine the behaviour of the local system near the puncture. Let us see that just on the local model: from (4.11) and (4.12), we deduce that, in the orthonormal frame \mathbf{e} ,

$$(5.1) \quad D = d + \left(s_i - \tau(s_i) - \text{Ad}(e^{\sqrt{-1}\theta\alpha_i}) \frac{Y_i + X_i}{\ln r^2} \right) \frac{dr}{r} + \sqrt{-1} \left(-\alpha_i + s_i + \tau(s_i) - \text{Ad}(e^{\sqrt{-1}\theta\alpha_i}) \frac{Y_i - H_i - X_i}{\ln r^2} \right) d\theta.$$

Observe that $Y_i - H_i - X_i = \text{Ad}(e^{-X_i})Y_i$ is nilpotent. The monodromy around x_i is

$$(5.2) \quad e^{2\pi\sqrt{-1}\alpha_i} e^{-s_i - \tau(s_i) + Y_i - H_i - X_i}.$$

In this formula the monodromy appears as the product of two commuting elements of G (the first is compact, the second is non compact). The $\frac{dr}{r}$ term has also an interpretation: taking

$$(5.3) \quad f = \mathbf{e} r^{-s_i + \tau(s_i)} (-\ln r^2)^{\frac{1}{2} \text{Ad}(\sqrt{-1}\theta\alpha_i)(X_i + Y_i)},$$

we get a G -trivialization, which is parallel along rays from the origin. The metric along these parallel rays has the form

$$(5.4) \quad h = r^{2(-s_i + \tau(s_i))} (-\ln r^2)^{\text{Ad}(\sqrt{-1}\theta\alpha_i)(X_i + Y_i)},$$

and in this trivialization the connection D takes the form

$$(5.5) \quad D = d - \sqrt{-1} \left(\alpha_i - s_i - \tau(s_i) + \text{Ad}(\sqrt{-1}\theta\alpha_i) \frac{Y_i - H_i - X_i}{2} \right) d\theta.$$

The logarithmic part of h in (5.4) gives no new information, because the triple (H_i, X_i, Y_i) is already encoded in the unipotent part $\exp(\sqrt{-1}\pi(Y_i - H_i - X_i))$ of the monodromy. But the semisimple part $s_i - \tau(s_i)$ is an additional information: it gives the polynomial order of growth of the harmonic metric h near the point x_i , along parallel rays. In the case $G = GL_n\mathbb{C}$, this additional structure transforms the local system into a **filtered local system** in the sense of Simpson, that is the fibre over the ray has a filtration with weights induced by $s_i - \tau(s_i)$.

In our case, the metric on the ray is a map to the symmetric space of non compact type G/H , and $\exp((s_i - \tau(s_i))u)$, where $u = -\ln r$, describes a geodesic in G/H , with some fixed speed, depending on s_i . The corresponding geometric data is a point on the geodesic boundary of G/H , and a positive real number describing the (constant) speed of the geodesic. This data is equivalent to that of a parabolic subgroup P of G with an antidominant character χ of the Lie algebra of P . This leads to the following definition.

Definition 5.1. A *parabolic G -local system on $X \setminus \{x_1, \dots, x_r\}$* is defined by the following data:

- (1) a G -local system F on $X \setminus \{x_1, \dots, x_r\}$,
- (2) on a ray ρ_i going to x_i , a choice of a parabolic subgroup P_i of $F(G)|_{\rho_i}$, invariant under the monodromy transformation around x_i , with a strictly antidominant character χ_i of the Lie algebra of P_i , where the $F_x(G)$ for $x \in \rho_i$ are identified by parallel transport.

Remark 5.2. Recall from Appendix D.1, that pairs (P, χ) consisting of a parabolic subgroup P of G and a strictly dominant character of the Lie algebra of \mathfrak{p} are in one-to-one correspondence with elements in \mathfrak{m} .

Remark 5.3. The invariance of the parabolic subgroup under the monodromy transformation makes the data independent of the choice of ρ_i .

Remark 5.4. Note that, in contrast with a parabolic bundle, at x_i the antidominant character is not constrained to lie in a Weyl alcove.

Let us come back to the G -local system coming from a G -Higgs bundle with Hermite-Einstein metric h . The formula (5.4) gives the behaviour of h for the model, but the formula remains valid for the adapted metric h_0 , up to lower order terms, and also for the Hermite-Einstein metric $h \in \mathcal{H}^\infty$, up to replacing h_0 by $h_0 e^s$, for some constant $s \in \mathfrak{t}_i \cap \mathfrak{t}'_i$. Thus we see that we have a well defined parabolic structure induced at the point x_i , with the parabolic subgroup and the character of its Lie algebra defined by $s_i - \tau(s_i)$. We have proved:

Proposition 5.5. *Let (E, ϕ) be a polystable parabolic G -Higgs bundle with $\text{pardeg}_\chi E = 0$ for all characters of G . Then the G -local system induced by the Hermite-Einstein metric constructed by Theorem 4.1 carries naturally a structure of parabolic G -local system.*

If $\alpha_i \in \sqrt{-1}\mathfrak{h}$ defines the parabolic structure of E at x_i , and $\text{Gr Res}_{x_i} \phi = s_i + Y_i$, then the parabolic structure is given by the element $s_i - \tau(s_i) \in \mathfrak{m}$, and the projection of the monodromy around x_i on the Levi group defined by $s_i - \tau(s_i)$ is

$$\exp(2\pi\sqrt{-1}\alpha_i) \exp(2\pi\sqrt{-1}(-s_i - \tau(s_i) + Y_i - H_i - X_i)).$$

To be precise, the compatibility between the metric and the parabolic structure is the following: on a ray going to x_i , we compare the metric h , seen as an application into G/H , with a geodesic given by the (Q_i, χ_i) , parametrized by $(-\ln r)$, and the condition is that the distance between them should grow at most like $|\ln r|^N$. For $G = \text{GL}(n, \mathbb{C})$, this is the property referred by Simpson in [42] as **tameness**.

Remark 5.6. If the $s_i - \tau(s_i)$ part of $\text{Gr Res}_{x_i} \phi$ vanishes, then we simply get a G -local system on $X - \{x_i\}$ with monodromy around x_i given by

$$\exp(2\pi\sqrt{-1}\alpha_i - s_i - \tau(s_i)) \exp(2\pi\sqrt{-1}(Y_i - H_i - X_i)).$$

Proof of proposition 5.5. We have already seen the model behaviour. In general we have a perturbed flat connection $D + a$, where D is the model (5.1) and the perturbation $a \in C^\infty_\delta$, which implies $a = a_r \frac{dr}{r} + a_\theta d\theta$, with $|a_r|, |a_\theta| = O(|\ln r|^{-1-\delta})$. The radial trivialization (5.3) is then modified by a bounded transformation (this does not change the parabolic weight $s_i - \tau(s_i)$ of the model), while the formula (5.5) for the connection in this radial trivialization comes with an additional term, depending only on the angle θ ,

$$b(\theta)d\theta = \text{Ad}(f^{-1})a_\theta d\theta.$$

The term $r^{-s_i + \tau(s_i)}$ in f and the initial bound on a_θ imply the vanishing of the components of b on the eigenspaces of $\text{ad}(s_i - \tau(s_i))$ corresponding to the nonnegative eigenvalues. On the contrary, there can be a nonzero contribution from the eigenspaces for the negative eigenvalues, which is an additional unipotent term in the monodromy, in the unipotent subgroup associated to the parabolic subgroup. Therefore, only the Levi part is fixed and is equal to that of the model. \square

The converse of Proposition 5.5 is given in the next section.

5.2. Harmonic metrics and polystability of G -parabolic local systems. Given a flat G -bundle (F, D) over X' and a metric h on F , that is a reduction of structure group to an H -bundle, we decompose $D = D_h^+ + \psi_h$, where D_h^+ is an H -connection and ψ_h is a section of $\Omega^1 \otimes E(\mathfrak{m})$. The metric h is said to be **harmonic** if

$$(5.6) \quad (D_h^+)^* \psi_h = 0.$$

If we regard the flat G -bundle as a representation $\rho : \pi_1(X') \rightarrow G$, then a harmonic metric is the same as a harmonic map from the universal cover of X' to

the symmetric space G/H , which is equivariant with respect to the action of the fundamental group on both sides.

A harmonic metric on a parabolic G -local system is a harmonic metric on the local system, which is tamed in the sense defined in the previous section. The existence of a harmonic metric on the a parabolic G -local system is governed, like for the Hermite-Einstein equation, by a stability condition. To define this we first define the degree. This is simpler than for G -Higgs bundles, since the global term of the degree vanishes here due to the flatness of the connection.

Let F be a parabolic G -local system with (P_i, χ_i) defining the parabolic structure at the point x_i . Let $Q \subset G$ be a parabolic subgroup of G and χ be an antidominant character of its Lie algebra \mathfrak{g} . Let σ be a reduction of structure group of F to a Q -bundle, which is constant (invariant under the flat connection). Using the relative degree of two parabolic subgroups with antidominant characters, we define the parabolic degree of F with respect to the reduction (P, χ) by the formula

$$(5.7) \quad \text{pardeg}(F)(P, \chi, \sigma) := - \sum_i \deg((P_i, \chi_i), (Q, \chi)).$$

This makes sense since both P_i and Q can be identified to subgroups of $F(G)$ near the marked point x_i .

We say that F is polystable if for any such reduction of the local system one has

$$\text{pardeg}(F)(P, \chi, \sigma) \geq 0,$$

with equality if and only if there is a reduction of the local system to a Levi subgroup of P . (When there is no parabolic structure, this condition only says that there is no reduction of the local system to a parabolic subgroup, unless there is a reduction to a Levi subgroup: the local system is reductive).

Theorem 5.7. *If F is a polystable parabolic G -local system, with vanishing parabolic degree with respect to all characters of \mathfrak{g} , then F admits a harmonic metric h (compatible with the parabolic structure near the marked points), which induces a polystable parabolic G -Higgs bundle. The relation between the structures at the marked points is the same as in Proposition 5.5.*

The proof of this theorem can be made formally similar to that of Theorem 4.1, see [43] and [7, §11], so we will not give the details of the proof of the existence of the harmonic metric.

The first step is to construct an initial metric h_0 (a section of $E(G/H)$) near a puncture x_i : start with a trivialization f where the flat connection D is given by formula (5.5), up to terms in the nilpotent part of the parabolic, and define the initial metric h_0 by formula (5.4). Implicit in this construction is the choice of appropriate Kostant-Sekiguchi \mathfrak{sl}_2 -triples (see Appendix § C). In the orthonormal trivialization $\mathbf{e} = fh_0^{-\frac{1}{2}}$, the flat connection has then the form (5.1), up to C_δ^∞ terms. Choose any extension of h_0 in the interior of X .

The second step is to define the functional space of metrics that we want to use: the relevant choice here is

$$(5.8) \quad \mathcal{H} = \{h = h_0 e^s, s \in \hat{L}_\delta^{2,p}(\mathfrak{m})\}.$$

As in §4.2, this space preserves the fact that h_0 can change at the points x_i , since $s(x_i) \in \mathfrak{t}_i \cap \mathfrak{t}'_i$. Of course, this is only a technical space needed for the proof, since at the end, we shall get by local elliptic regularity, as in (4.20),

$$(5.9) \quad h \in \mathcal{H}^\infty = \{h_0 e^s, s \in \hat{C}_\delta^\infty(\mathfrak{m})\}.$$

To solve it, one minimizes the energy

$$\int_X (|\psi_h|^2 - |\psi_{h_0}|^2)$$

on \mathcal{H} , under the constraint $\|(D_h^+)^*\psi_h\|_{L_\delta^p} \leq B$. The heart of the proof consists in proving that non convergence of a minimizing sequence would imply the existence of a reduction of E to a parabolic subgroup P , appearing with an antidominant character, which would contradict stability.

The harmonic metric h induces a G -Higgs bundle (E, ϕ) , where the $\bar{\partial}$ -operator of E is $(D^+)^{0,1}$ and the Higgs field $\phi = \psi_h^{1,0}$. A priori, this defines (E, ϕ) only in the interior of X , and we have to extend it over the points x_i . Fortunately, because of the good control (5.9) on the solution, this can be done relatively easily. Because $h \in \mathcal{H}^\infty$, we have, in an orthonormal trivialization \mathbf{e} ,

$$\begin{aligned} \bar{\partial}^E &= \bar{\partial}_0 + a, & \bar{\partial}_0 &= \bar{\partial} + \left(\alpha_i - \frac{H_i}{\ln|z|^2}\right) \frac{d\bar{z}}{2\bar{z}}, \\ \phi &= \phi_0 + b, & \phi_0 &= \left(s_i - \frac{\text{Ad}(e^{\sqrt{-1}\theta}\alpha_i)Y_i}{\ln|z|^2}\right) \frac{dz}{z}, \end{aligned}$$

where the perturbations a and b belong to the space C_δ^∞ .

Lemma 5.8. *There exists a gauge transformation $g \in \hat{C}_\delta^\infty(E(H^\mathbb{C}))$, defined in a neighbourhood of x_i , such that*

$$g(\bar{\partial}^E) = \bar{\partial}_0.$$

Proof. The equation to solve is

$$g^{-1}\bar{\partial}_0 g = \text{Ad}(g^{-1})a.$$

Writing $g = \exp(u)$, the equation becomes $e^{-u}\bar{\partial}_0 e^u = e^{-\text{ad } u}a$, with linear part $\bar{\partial}_0 u = a$. One can deduce an explicit solution for the linear problem from the Cauchy kernel, with suitable estimates [7, Lemma 9.1]. Shrinking to a smaller ball if necessary, a fixed point argument gives the expected solution. \square

We now have a \hat{C}_δ^∞ -gauge $f = \mathbf{e}g$ of \mathbf{E} (seen as a $H^\mathbb{C}$ -bundle) in which $\bar{\partial}^E$ is exactly the model $\bar{\partial}_0$, and $\phi = \phi_0 + b'$ for some $b' \in C_\delta^\infty$. We deduce an explicit holomorphic gauge,

$$e = f r^{-\alpha_i} (-\ln r^2)^{H_i/2},$$

which we use to extend the holomorphic bundle E over x_i . Moreover, in the gauge e , the Higgs field becomes

$$\phi = \text{Ad}(r^{\alpha_i} (-\ln r^2)^{-H_i/2})(\phi_0 + b').$$

Here

$$\text{Ad}(r^{\alpha_i} (-\ln r^2)^{-H_i/2})\phi_0 = (s_i + \text{Ad}(z^{\alpha_i})Y_i) \frac{dz}{z}$$

is just our model for the Higgs field in the holomorphic trivialization e , with $\text{Gr Res}_{x_i} = s_i + Y_i$, so we have to analyse the behaviour of the remainder

$$\phi' = \text{Ad}(r^{\alpha_i} (-\ln r^2)^{-H_i/2})b', \quad b' = O\left(\frac{1}{|\ln r|^\delta}\right) \frac{dz}{z \ln r}.$$

Here remind that b' is holomorphic outside the origin, and $0 < \delta < 1$. This implies that ϕ' is meromorphic: decompose

$$\phi' = \sum_\mu \phi'_\mu \frac{dz}{z}$$

along the eigenvalues μ of $\text{ad}(\alpha_i)$ on $\mathfrak{m}^{\mathbb{C}}$, and let us analyse the pole of ϕ'_{μ} . We certainly have

$$\phi'_{\mu} = O(r^{\mu} |\ln r|^N)$$

which implies $v(\phi'_{\mu}) \geq -\lfloor -\mu \rfloor$, as wanted for a parabolic G -Higgs bundle. We can say something more on $\text{Gr Res}_{x_i} \phi'$, that is on the components ϕ'_{μ} for $\mu \in \mathbb{Z}$: decompose further with respect to the eigenvalues η of $\text{ad}(H_i)$ on $\mathfrak{m}^{\mathbb{C}}$, and we get

$$\phi'_{\mu, \eta} = O(r^{\mu} |\ln r|^{-\frac{\eta}{2} - 1 - \delta}).$$

This implies that the (μ, η) -component for $\mu \in \mathbb{Z}$ can be non zero only if $\eta < -2$, which implies that $Y_i + \text{Gr Res}_{x_i} \phi'$ is conjugate to Y_i , that is $\text{Gr Res}_{x_i} \phi$ remains conjugate to $s_i + Y_i$. This finishes the proof of theorem 5.7. \square

We give below the table of the relations between the singularities for the Higgs bundle and for the corresponding local systems, similar to Simpson's table in [42]. The interesting feature here is that they correspond in a way which extends the Kostant-Sekiguchi correspondence, see appendix C. More specifically, in the nilpotent case, one gets exactly the correspondence between $H^{\mathbb{C}}$ -nilpotent orbits in $\mathfrak{m}^{\mathbb{C}}$ and nilpotent G -orbits in \mathfrak{g} which is the Kostant-Sekiguchi correspondence (this was turned by Vergne [45] into a diffeomorphism which can be seen as a toy model of our correspondence theorem). The more general case with semisimple residues corresponds to an extension of Kostant-Sekiguchi-Vergne.

	Weight	Monodromy (projected in the Levi)
(E, ϕ)	α	$s + Y = \text{Gr Res}_x \phi$
(F, ∇)	$s - \tau(s)$	$\exp(2\pi\sqrt{-1}\alpha) \exp(2\pi\sqrt{-1}(-s - \tau(s) + Y - H - X))$

Table of relations for weights and monodromies

APPENDIX A. WEYL ALCOVES AND CONJUGACY CLASSES OF A COMPACT LIE GROUP

For the following see e.g. [15].

Let H be a compact Lie group with Lie algebra \mathfrak{h} . Let $\langle \cdot, \cdot \rangle$ be a H -invariant inner product on \mathfrak{h} . Let $T \subset H$ be a Cartan subgroup, i.e. a maximal torus, and $\mathfrak{t} \subset \mathfrak{h}$ be its Lie algebra (a Cartan subalgebra). After fixing a system of simple roots we choose positive roots R_+ . Consider the family of affine hyperplanes in \mathfrak{t}

$$\mathcal{H}_{\lambda n} = \lambda^{-1}(n), \quad \lambda \in R^+, \quad n \in \mathbb{Z}$$

together with the union $\mathfrak{t}_s = \cup_{\lambda, n} \mathcal{H}_{\lambda n}$. This family is given by the inverse image of the set of singular points of T under the exponential map

$$(A.1) \quad \exp : \mathfrak{t} \longrightarrow T.$$

The set $\mathfrak{t} - \mathfrak{t}_s$ decomposes into convex connected components which are called the **alcoves** of H (sometimes also referred as **Weyl alcoves**). Note that, by definition, alcoves are open. A **wall** of an alcove \mathcal{A} are those subsets of $\bar{\mathcal{A}} \cap \mathcal{H}_{\lambda, n}$ of \mathfrak{t} that have dimension $k - 1$, where $\bar{\mathcal{A}}$ is the closure of \mathcal{A} and $k = \text{rank}(H)$.

Let $W := N(T)/T$ be the Weyl group of H . The H -invariant inner product on \mathfrak{h} induces a W -invariant inner product in \mathfrak{t} . The **co-character lattice** $\Lambda_{\text{cochar}} \subset \mathfrak{t}$ is defined as the kernel of the exponential map (A.1). Recall that the **co-roots** are the elements of \mathfrak{t} defined by

$$\lambda^* = 2b^{-1}(\lambda)/\langle \lambda, \lambda \rangle,$$

where b is the isomorphism $b : \mathfrak{t} \longrightarrow \mathfrak{t}^*$ defined by the inner product $\langle \cdot, \cdot \rangle$.

The co-roots define a lattice $\Lambda_{\text{coroot}} \subset \mathfrak{t}$. We have that $\Lambda_{\text{coroot}} \subset \Lambda_{\text{cochar}}$ and $\pi_1(H) = \Lambda_{\text{cochar}}/\Lambda_{\text{coroot}}$. In particular, $\Lambda_{\text{coroot}} = \Lambda_{\text{cochar}}$ if H is simply connected.

The **affine Weyl group** is defined as

$$W_{\text{aff}} = \Lambda_{\text{cochar}} \rtimes W$$

where Λ_{cochar} acts on \mathfrak{t} by translations. We have the following.

Proposition A.1. *The closure $\bar{\mathcal{A}}$ of any alcove \mathcal{A} is a fundamental domain for the action of W_{aff} on \mathfrak{t} , i.e. every W_{aff} -orbit meets $\bar{\mathcal{A}}$ in exactly one point.*

The following will be very important for our definition of parabolic Higgs bundle and for the analysis involved in the Hitchin-Kobayashi correspondence.

Proposition A.2. *Let \mathcal{A} be an alcove of H containing $0 \in \mathfrak{t}$. Then:*

(i) *For $\alpha \in \bar{\mathcal{A}}$ every eigenvalue λ of the linear map $\text{ad}(\alpha) : \mathfrak{h} \rightarrow \mathfrak{h}$ satisfies $|\lambda| \leq 1$. Moreover, if we define*

$$\mathcal{A}' = \{\alpha \in \bar{\mathcal{A}} : |\lambda| < 1, \text{ for every eigenvalue } \lambda \text{ of } \text{ad} \alpha\},$$

we have $\mathcal{A} \subset \mathcal{A}'$.

(ii) *If H is simple and λ_0 is the highest root, then*

$$\mathcal{A}' := \{\alpha \in \bar{\mathcal{A}} \mid \lambda_0(\alpha) < 1\}, \quad \text{and} \quad \bar{\mathcal{A}} \setminus \mathcal{A}' = \{\alpha \in \bar{\mathcal{A}} \mid \lambda_0(\alpha) = 1\}.$$

(iii) *For any $\alpha \in \bar{\mathcal{A}}$ there exist $p \in \mathbb{N}$ and $w \in W_{\text{aff}}$ such that $w(p\alpha)$ is as close to 0 as we wish, in particular it is in \mathcal{A}' .*

The alcoves of H are important for us because of their relation with conjugacy classes of H . If H is connected, every element of H is conjugate to an element of T , in particular every element of H lies in a Cartan subgroup. If $\text{Conj}(H)$ is the space of **conjugacy classes** of H we have then homeomorphisms

$$\text{Conj}(H) \simeq T/W \simeq \mathfrak{t}/W_{\text{aff}}.$$

From this and Proposition A.1, we have the following.

Proposition A.3. *Let H be a connected compact Lie group. Then the space of conjugacy classes of H is homeomorphic to $\bar{\mathcal{A}}$.*

Proposition A.4. *Let G be the complexification of a connected compact Lie group H , and let $T^{\mathbb{C}}$ be the complexification of a Cartan subgroup of H . Then every element of G is conjugate to an element of $T^{\mathbb{C}}$, which in particular can be written as $\exp(\alpha)\exp(s)$ with $\alpha \in \bar{\mathcal{A}}$, $s \in i\mathfrak{t}$ and $[\alpha, s] = 0$.*

APPENDIX B. REAL REDUCTIVE LIE GROUPS

Following Knapp [27, p. 384], a **real reductive Lie group** is defined as a 4-tuple (G, H, θ, B) , where G is a real Lie group, $H \subset G$ is a maximal compact subgroup, $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ is a Cartan involution, and B is a non-degenerate bilinear form on \mathfrak{g} , which is $\text{Ad}(G)$ -invariant and θ -invariant. The data (G, H, θ, B) has to satisfy in addition that

- the Lie algebra \mathfrak{g} of G is reductive
- θ gives a decomposition (the Cartan decomposition)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

into its ± 1 -eigenspaces, where \mathfrak{h} is the Lie algebras of H , so we have

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h},$$

- \mathfrak{h} and \mathfrak{m} are orthogonal under B , and B is positive definite on \mathfrak{m} and negative definite on \mathfrak{h} ,
- multiplication as a map from $H \times \exp \mathfrak{m}$ into G is an onto diffeomorphism.
- Every automorphism $\text{Ad}(g)$ of $\mathfrak{g}^{\mathbb{C}}$ is inner for $g \in G$, i.e., is given by some x in Intg .

Of course a compact real Lie group G whose Lie algebra is equipped with a non-degenerate $\text{Ad}(G)$ -invariant bilinear form is reductive in the sense defined above. Also the underlying real structure of a the complexification $G^\mathbb{C}$ of a compact Lie group H , whose Lie algebra \mathfrak{h} is equipped with a non-degenerate $\text{Ad}(H)$ -invariant bilinear form can be endowed with a natural reductive structure.

If G is semisimple, the data (G, H, θ, B) can be recovered (to be precise, the quadratic form B can only recovered up to a scalar but this will be sufficient for everything we do in this paper) from the choice of a maximal compact subgroup $H \subset G$. There are other situations where less information does the job, e.g. for certain linear groups (see [27, p. 385]).

The bilinear form B does not play any role in the definition of a parabolic G -Higgs bundle but is essential for defining the stability condition and the gauge equations involved in the Hitchin-Kobayashi correspondence.

Note that the compactness of H together with the last property above say that G has only finitely many components.

Let $\mathfrak{g}^\mathbb{C}$ and $\mathfrak{h}^\mathbb{C}$ be the complexifications of \mathfrak{g} and \mathfrak{h} respectively, and let $H^\mathbb{C}$ be the complexification of H . Let

$$(B.1) \quad \mathfrak{g}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \mathfrak{m}^\mathbb{C}$$

be the complexification of the Cartan decomposition. The group H acts linearly on \mathfrak{m} through the adjoint representation, and this action extends to a linear holomorphic action of $H^\mathbb{C}$ on $\mathfrak{m}^\mathbb{C} = \mathfrak{m} \otimes \mathbb{C}$, the **isotropy representation** that we will denote by

$$\iota : H^\mathbb{C} \longrightarrow \text{Aut}(\mathfrak{m}^\mathbb{C}),$$

which we will denote sometimes by Ad since it is obtained by restriction of the adjoint representation of G .

If G is complex with maximal compact subgroup H , then $\mathfrak{g} = \mathfrak{h} \oplus i\mathfrak{h}$. We thus have that $\mathfrak{m} = i\mathfrak{h}$, and the isotropy representation coincides with the adjoint representation $\text{Ad} : G \longrightarrow \text{Aut}(\mathfrak{g})$

If G is a reductive group, then the map $\Theta : G \longrightarrow G$ defined by

$$(B.2) \quad \Theta(h \exp A) = h \exp(-A) \quad \text{for } h \in H \text{ and } A \in \mathfrak{m}$$

is an automorphism of G and its differential is θ . This is called the **global Cartan involution**.

APPENDIX C. \mathfrak{sl}_2 -TRIPLES AND ORBIT THEORY

We consider a reductive group G as defined in Appendix B.

An ordered triple of elements (x, e, f) in \mathfrak{g} (or $\mathfrak{g}^\mathbb{C}$) is called a **\mathfrak{sl}_2 -triple** if the bracket relations $[x, e] = 2e$, $[x, f] = -2f$, and $[e, f] = x$ are satisfied. One has that the elements e and f are nilpotent. A \mathfrak{sl}_2 -triple (x, e, f) in $\mathfrak{g}^\mathbb{C}$ is called **normal** if $e, f \in \mathfrak{m}^\mathbb{C}$ and $x \in \mathfrak{h}^\mathbb{C}$. Some times we refer to a normal triple as a Kostant-Rallis triple (see [28]).

Let $\mathcal{N}(\mathfrak{g})$ and $\mathcal{N}(\mathfrak{m}^\mathbb{C})$ be the set of nilpotent elements in \mathfrak{g} and $\mathfrak{m}^\mathbb{C}$ respectively. One has the following (see Kostant-Rallis [28]).

Proposition C.1. (1) Every element $0 \neq e \in \mathcal{N}(\mathfrak{g})$ can be embedded in a $\mathfrak{sl}(2)$ -triple (x, e, f) , establishing a correspondence between the set of all G -orbits in $\mathcal{N}(\mathfrak{m}^\mathbb{C})$ and the set of all G -conjugacy classes of $\mathfrak{sl}(2)$ -triples in \mathfrak{g} .

(2) Every element $0 \neq e \in \mathcal{N}(\mathfrak{m}^\mathbb{C})$ can be embedded in a normal $\mathfrak{sl}(2)$ -triple (x, e, f) , establishing a correspondence between the set of all $H^\mathbb{C}$ -orbits in $\mathcal{N}(\mathfrak{m}^\mathbb{C})$ and the set of all $H^\mathbb{C}$ -conjugacy classes of normal $\mathfrak{sl}(2)$ -triples $\mathfrak{g}^\mathbb{C}$.

We say that an \mathfrak{sl}_2 -triple in \mathfrak{g} is a Kostant-Sekiguchi triple if $\theta(e) = -f$, and hence $\theta(x) = -x$. A normal \mathfrak{sl}_2 -triple in $\mathfrak{g}^{\mathbb{C}}$ is called Kostant-Sekiguchi triple if $f = \sigma(e)$ where σ is the conjugation of $\mathfrak{g}^{\mathbb{C}}$ defining \mathfrak{g} .

One has the following (see [40]).

Proposition C.2. (1) Every \mathfrak{sl}_2 -triple in \mathfrak{g} is conjugate under G to a Kostant-Sekiguchi triple in \mathfrak{g} . Two Kostant-Sekiguchi triples are conjugate under G to the same \mathfrak{sl}_2 -triple in \mathfrak{g} if and only if they are conjugate under H .

(2) Every Kostant-Rallis triple in $\mathfrak{g}^{\mathbb{C}}$ is conjugate under $H^{\mathbb{C}}$ to a Kostant-Sekiguchi triple. Two Kostant-Sekiguchi triples are conjugate under $H^{\mathbb{C}}$ to the same Kostant-Rallis triple if and only if they are conjugate under H .

Propositions C.2 and C.1 can be combined to obtain the Kostant-Sekiguchi correspondence:

Proposition C.3. There is a one-to-one correspondence (see [40, 45]).

$$\mathcal{N}(\mathfrak{g})/G \longleftrightarrow \mathcal{N}(\mathfrak{m}^{\mathbb{C}})/H^{\mathbb{C}}.$$

A similar correspondence can be established for orbits of arbitrary elements (see [4, 8]).

APPENDIX D. PARABOLIC SUBGROUPS AND RELATIVE DEGREE

In this section we recall some basics on parabolic subgroups and define precisely the relative degree of two parabolic subgroups.

D.1. Parabolic subgroups. Let $\Sigma = H \backslash G$ a symmetric space of non compact type. (The action is taken to be a right action, because this fits better with the way symmetric spaces arise in Kähler quotients). Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the Cartan decomposition, and $\mathfrak{a} \subset \mathfrak{m}$ a maximal abelian subalgebra (the dimension of \mathfrak{a} is the rank of Σ). Let $\Phi \subset \mathfrak{a}^*$ the roots of \mathfrak{g} , so that $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}_{\lambda}$, where \mathfrak{g}_0 is the centralizer of \mathfrak{a} .

Choose a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and let $\Phi^{\pm} \subset \Phi$ (resp. $\Delta \subset \Phi$) be the set of positive/negative roots (resp. simple roots). For $I \subset \Delta$, denote $\Phi^I \subset \Phi$ the set of roots which are linear combinations of elements of I , then we define a standard parabolic subalgebra

$$\mathfrak{p}_I = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Phi^I \cup \Phi^-} \mathfrak{g}_{\lambda},$$

and $P_I \subset G$ the corresponding subgroup. Any parabolic subalgebra of \mathfrak{g} is conjugate to one of the standard parabolic subalgebras.

Given an element $s \in \mathfrak{a}^+$, one picks a standard parabolic subgroup P_s in the following way. When $t \rightarrow +\infty$, the geodesic $t \mapsto *e^{ts}$ in Σ (where $*$ is the base point, fixed by H), hits the visual boundary $\partial_{\infty}\Sigma$ in a point, whose stabilizer in G is the parabolic group P_s . It follows that P_s and \mathfrak{p}_s are defined by

$$\begin{aligned} P_s &= \{g \in G, e^{ts}ge^{-ts} \text{ is bounded as } t \rightarrow \infty\}, \\ \mathfrak{p}_s &= \{x \in \mathfrak{g}, \text{Ad}(e^{ts})x \text{ is bounded as } t \rightarrow \infty\}. \end{aligned}$$

It is immediate that

$$\mathfrak{p}_s = \sum_{\lambda \in R, \lambda(s) \leq 0} \mathfrak{g}_{\lambda} = \mathfrak{p}_I$$

for $I = \{\lambda \in \Delta, \lambda(s) = 0\}$. The element s also defines a Levi subgroup $L_s \subset P_s$ and a Levi subalgebra $\mathfrak{l}_s \subset \mathfrak{p}_s$ by

$$L_s = \{g \in G, gsg^{-1} = s\}, \quad \mathfrak{l}_s = \{x \in \mathfrak{g}, [s, x] = 0\}.$$

Moreover, the mapping $\chi : \mathfrak{p} \rightarrow \mathbb{R}$ defined by

$$\chi_s(x) = \langle s, x \rangle$$

is a **strictly antidominant character** of \mathfrak{p}_s .

D.2. Relative degree. We now define a function which is important in the paper, since it calculates the contribution to the parabolic degree at the punctures. The setting is the following.

Let $\mathcal{O}_H \subset \mathfrak{m}$ an H -orbit in \mathfrak{m} . As is well known, \mathcal{O}_H is also a G -homogeneous space. This can be seen in the following way: given $s \in \mathcal{O}_H$, one can consider $\eta(s) = \lim_{t \rightarrow +\infty} *e^{ts} \in \partial_\infty \Sigma$. It turns out that the image of \mathcal{O}_H under η is a G -orbit in $\partial_\infty \Sigma$. Of course the stabilizer of $\eta(s)$ is the parabolic group P_s defined above, so one gets an identification

$$\eta : \mathcal{O}_H = H/(P_s \cap H) \subset \mathfrak{m} \longrightarrow P_s \backslash G \subset \partial_\infty \Sigma.$$

The action of $g \in G$ on \mathcal{O}_H will be denoted by $s \cdot g$; if one decomposes $g = ph$ with $h \in H$ and $p \in P$, then $s \cdot g = s \cdot h = \text{Ad}(h^{-1})s$.

Now take another element $\sigma \in \mathfrak{m}$. As we shall see below in the proof of the proposition, the function

$$t \mapsto \langle s \cdot e^{-t\sigma}, \sigma \rangle$$

is a nonincreasing function of t , so we can define a function

$$\mu_s : \mathfrak{m} \longrightarrow \mathbb{R}, \quad \mu_s(\sigma) = \lim_{t \rightarrow +\infty} \langle s \cdot e^{-t\sigma}, \sigma \rangle.$$

This function is actually (up to a normalization) a function defined on $\partial_\infty \Sigma$, as follows from the following proposition.

Proposition D.1. *Suppose s and σ normalized so that $|s| = |\sigma| = 1$. Then one has*

$$\mu_s(\sigma) = \cos \angle_{\text{Tits}}(\eta(\sigma), \eta(s)),$$

where \angle_{Tits} is the Tits distance on $\partial_\infty \Sigma$. In particular, one has the reciprocity $\mu_\sigma(s) = \mu_s(\sigma)$.

Proof. Decompose $e^{-t\sigma} = p_t h_t$ with $h_t \in H$ and $p_t \in P_s$. Then

$$\langle s \cdot e^{-t\sigma}, \sigma \rangle = \langle \text{Ad}(h_t^{-1})s, \sigma \rangle = \cos \angle(\text{Ad}(h_t^{-1})s, \sigma).$$

On the other hand, the distance

$$d(*e^{u \text{Ad}(h_t^{-1})s} e^{t\sigma}, *e^{us}) = d(*e^{us} p_t^{-1}, *e^{us})$$

is bounded when $u \rightarrow +\infty$, so $u \rightarrow *e^{u \text{Ad}(h_t)s} e^{t\sigma}$ is the geodesic emanating from $*e^{t\sigma}$ and going to $\eta(s)$. So the angle between the geodesics emanating from $*e^{t\sigma}$ and converging to $\eta(\sigma)$ and $\eta(s)$ is the angle between $\text{Ad}(h_t^{-1})s$ and σ . It is well known that this angle is increasing and converges when $t \rightarrow \infty$ to the Tits distance between $\eta(s)$ and $\eta(\sigma)$, and the proposition follows. \square

The function $-\mu_\sigma$ is the ‘asymptotic slope’ of [26]. In [35], the complex case is studied: if $G = H^\mathbb{C}$ then the adjoint orbit \mathcal{O}_H is a Kähler manifold, and the asymptotic slope can be reinterpreted in terms of maximal weights of the action of G on \mathcal{O}_H .

We will use this notion to define a relative degree. From the proposition, $\mu_s(\sigma)$ depends only on giving two pairs (P, s) and (Q, σ) of a parabolic subgroup of G and an antidominant character on the parabolic subgroup. So we can define the **relative degree** of (P, s) and (Q, σ) as

$$(D.1) \quad \deg((P, s), (Q, \sigma)) = \mu_s(\sigma).$$

Observe, again from the proposition, that \deg is a symmetric function of its two arguments.

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